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Stabilized mixed hp-BEM for frictional contact problems in linear elasticity

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Abstract

We analyze stabilized mixed hp-boundary element methods for frictional contact problems for the Lamé equation. The stabilization technique circumvents the discrete inf-sup condition for the mixed problem and thus allows us to use the same mesh and polynomial degree for the primal and dual variables. We prove a priori convergence rates in the case of Tresca friction, using Gauss-Legendre-Lagrange polynomials as test and trial functions for the Lagrange multiplier. Additionally, a residual based a posteriori error estimate for a more general class of discretizations is derived. It in particular applies to discretizations based on Bernstein polynomials for the discrete Lagrange multiplier, which we also analyze. The discretization and the a posteriori error estimate are extended to the case of Coulomb friction. Several numerical experiments underline our theoretical results, demonstrate the behavior of the method and its insensitivity to the scaling and perturbations of the stabilization term.

Keywords: Tresca/Coulomb frictional contact problem, stabilized hp-BEM, a priori and a posteriori error estimates

1. Introduction

Mechanical problems naturally involve frictional contact with surrounding materials. The frictional contact problems studied in this article consist of a differential equation balancing the forces within the object at hand and contact and friction constraints on one part of the object's boundary. The latter significantly complicate the numerical analysis and computations as they give rise to a variational inequality with a closed, convex set K of admissible test and trail functions and a non-differentiable functional for the frictional energy, $j(\cdot)$, see e.g. [14, 21, 24].
With the help of a Lagrange multiplier λ, which represents the negative surface traction on the contact boundary, the variational inequality can be formulated as a mixed problem, such that the constraint from the variational inequality takes a simpler form. If one directly discretizes the mixed problem, a discrete inf-sup condition is required to obtain a unique discrete solution [21]. Even if the inf-sup condition is satisfied, the possible dependence of the discrete inf-sup constant on the mesh size h and polynomial degree p affects the convergence rate of the numerical method and must be known to derive a priori error estimates. Sufficient coarsening of the mesh size and reducing the polynomial degree for the discretization of the Lagrange multiplier λ guarantees a uniform bound for the discrete inf-sup constant [31]. What sufficient means explicitly remains open, though a doubled mesh size $k = 2h$ and a polynomial degree reduced by one $q = p - 1$ are found to work in practice [31]. In particular, the discrete inf-sup condition is not satisfied for the same mesh size $k = h$ and polynomial degree $q = p$, which which would significantly simplify the data structure and the computations. For finite elements it is well known that the discrete inf-sup condition in mixed formulations can be circumvented by introducing a stabilization term [4, 5]. In this article we consider the stabilization of mixed hp-boundary element methods, their a priori and a posteriori error analysis and validation in numerical experiments. The stabilized procedures are constructed from an equivalent saddle point problem which is strictly concave in the

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second, dual variable (see Theorem 7).
The challenges of the friction and contact constraints do not only involve the formulation, but also the observed error reduction: Typically, the solution is of reduced regularity at the interface from contact to non-contact and from stick to slip condition. The location of these interfaces is a priori unknown, so that special meshes like geometrically graded ones cannot be applied. An a posteriori error estimate with an automatic mesh refinement procedure is required to resolve the singularities. As a drawback, the adaptive methods require to compute an entire sequence of solutions, rather than one solution only. For computational advantages the convergence rate of the adaptive method should be significantly higher than the one of the quasi-uniform method. For example, hp-adaptivity is well suited to achieve this [3]. There, a non-stabilized mixed BEM formulation is analyzed, which relies on special basis functions, namely Gauss-Lobatto-Lagrange for the primal variable and its biorthogonal counterpart for the dual variable. The unproven p-dependency of the discrete inf-sup constant for biorthogonal basis functions is here circumvented by an appropriate stabilization term. Moreover, no special basis functions are needed.
In many cases the insufficient resolution of these interfaces is the dominant source of error [3]. As they lie on the boundary only, it seems to be favorable to reduce the differential problem to the boundary as in [17, 16], and use a boundary element method. Thereby one only requires a boundary mesh rather than precise refinements (both h and p) on the trace mesh induced by refinements of a mesh in the domain. As a drawback of the boundary element methods, compared to FEM, the system matrix is densely populated and the computation of the entries requires the evaluation of singular integrals. We refer to [29] where several strategies to overcome these BEM specific difficulties are discussed.

Most of the arguments in our article carry over to the 3d problem, at least for rectangular meshes. The functional analytic parts hold verbatim, and for rectangular meshes one can consider tensor product discretizations of the Lagrange multiplier. However, a number of new technical and notational issues arise, e.g. the assumption of Lemma 14 might not be satisfied even for fine meshes in 3d and, computationally, adaptive mesh refinements are restricted to rectangular meshes. We therefore restrict ourselves to 2d.

The paper is structured as follows. In Section 2 we introduce a mixed boundary element method with the help of the Poincaré-Steklov operator S, which maps the displacement u on the boundary to the boundary traction $\sigma(u)n = -\lambda$. The existence of a unique solution (u, λ) of the mixed formulation of the original Tresca friction contact problem is based on the coercivity of the underlying bilinear form $\langle S\cdot, \cdot\rangle$ on the trace space $\widetilde{H}^{1/2}(\Gamma_\Sigma)$ on the Neumann and friction part of the boundary, as well as the inf-sup condition for λ in the dual space (see Theorem 1). In Section 3 we discretize the mixed formulation in suitable piecewise polynomial subspaces. On a locally quasi-uniform mesh we use linear combinations of affinely transformed Bernstein polynomials or Gauss-Legendre-Lagrange polynomials for the Lagrange multiplier. In both cases we impose additional conditions to reflect the constraints of non-penetration and stick-slip in the original contact problem. Based on these hp-boundary element spaces we introduce a stabilized mixed method (13) with stabilization parameter $\gamma|_E$, which depends on the mesh size and polynomial degree on the element E of the subdivision $\mathcal{T}_h$ of Γ_Σ. As in [20] for the h-version FEM, the stabilized discrete mixed scheme admits a unique solution (u^{hp}, λ^{kq}) (Theorem 7). We derive a priori error estimates for the Galerkin error in the displacement u and the Lagrange multiplier λ which are explicit in the polynomial degrees p, q, see Section 4. Our results (Theorem 16 and Remark 18) cannot fully recover the FEM result for the lowest order h-version in [18, 19] due to the approximation of the Poincaré-Steklov operator in the stabilization term. In Section 5 we derive an a posteriori error estimate of residual type for a general class of Lagrange multiplier discretizations (Theorem 22). After discussing implementational challenges in Section 6, we give an extension of our approach to Coulomb friction in Section 7 by suitably modifying the test and ansatz spaces. Finally, our numerical experiments in Section 8 underline our theoretical results, demonstrate the behavior of the method and its insensitivity to the scaling and perturbations of the stabilization term. They clearly show that the classical stabilization technique extends to variational inequality problems, here for contact problems, handled with boundary integral equations and hp-methods.

Notation: C, C' or $\widetilde{C}$ denote generic, positive constants which may take different values at different positions. These constants may dependent on the material parameters, the domain, especially the Dirichlet boundary and the shape regularity of the mesh, but they are independent of the quantities of interest, namely the mesh sizes and polynomial degrees.

2. A mixed boundary integral formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary Γ and outward unit normal n. We assume that Γ is already sufficiently scaled such that $\operatorname{cap}(\Gamma) < 1$. Furthermore, let $\overline{\Gamma} = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_C$ be decomposed into non-overlapping, for simplicity connected, Dirichlet, Neumann and contact boundary parts, and denote by $\overline{\Gamma}_\Sigma := \overline{\Gamma}_N \cup \overline{\Gamma}_C$ the union of the latter two. For the ease of presentation regarding the correct dual space for the contact stresses we assume $\overline{\Gamma}_D \cap \overline{\Gamma}_C = \emptyset$. For given gap function $g \in H^{1/2}(\Gamma_C)$, friction threshold $0 < \mathcal{F} \in L^2(\Gamma_C)$, Neumann data $f \in H^{-1/2}(\Gamma_N)$ and elasticity tensor C the considered Tresca frictional contact problem is to find a function $u \in H^1_{\Gamma_D}(\Omega) := \left\{ v \in H^1(\Omega) : \; v|_{\Gamma_D} = 0 \right\}$ such that

$$
\begin{aligned}
-\operatorname{div} \sigma(u) &= 0 && \text{in } \Omega && \text{(1a)}\\
\sigma(u) &= C : \epsilon(u) && \text{in } \Omega && \text{(1b)}\\
u &= 0 && \text{on } \Gamma_D && \text{(1c)}\\
\sigma(u)n &= f && \text{on } \Gamma_N && \text{(1d)}\\
\sigma_n \leq 0, \; u_n \leq g, \; \sigma_n(u_n - g) &= 0 && \text{on } \Gamma_C && \text{(1e)}\\
|\sigma_t| \leq \mathcal{F}, \; \sigma_t u_t + \mathcal{F}\,|u_t| &= 0 && \text{on } \Gamma_C. && \text{(1f)}
\end{aligned}
$$

Here, σ_n, u_n, σ_t, $u_t \in \mathbb{R}$ are the normal, tangential components of $\sigma(u)n$, u, respectively and (1b) describes Hooke's law with the linearized strain tensor $\epsilon(u) = \frac{1}{2}\left(\nabla u + \nabla u^\top\right)$. Equation (1f) may equivalently be written in the form

$$
|\sigma_t| \leq \mathcal{F}, \qquad |\sigma_t| < \mathcal{F} \Rightarrow u_t = 0, \qquad |\sigma_t| = \mathcal{F} \Rightarrow \exists \alpha \geq 0 : u_t = -\alpha\sigma_t. \tag{2}
$$

Testing (1) with $v_\Omega \in K_\Omega := \left\{ v_\Omega \in H^1_{\Gamma_D}(\Omega) : (v_\Omega)_n \leq g \text{ a.e. on } \Gamma_C \right\}$ and introducing the friction functional $j(v) := \int_{\Gamma_C} \mathcal{F}\,|v_t|\,ds$ yields the (domain) variational inequality formulation:

$$
u_\Omega \in K_\Omega : \; (\sigma(u_\Omega), \epsilon(v_\Omega - u_\Omega))_{0,\Omega} + j(v_\Omega) - j(u_\Omega) \geq \langle f, v_\Omega - u_\Omega \rangle_{\Gamma_N} \quad \forall v_\Omega \in K_\Omega \,, \tag{3}
$$

where $(u, v)_{0,\Omega} = \int_\Omega uv\,dx$ and $\langle f, v \rangle_{\Gamma_N} = \int_{\Gamma_N} fv\,ds$ are defined by duality.

Boundary integral formulations can be advantageous for problems with non-linear boundary conditions and with no source terms in Ω. They rely on the explicit formula for the fundamental solution of the Lamé equation in $\mathbb{R}^2$:

$$
G(x,y) = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} \left(\log|x - y| I + \frac{\lambda + \mu}{\lambda + 3\mu} \frac{(x-y)(x-y)^\top}{|x-y|^2} \right).
$$

With the help of the traction operator $(\mathbb{T}u)_i = \lambda n_i \operatorname{div} u + \mu \partial_n u_i + \mu \left\langle \frac{\partial u}{\partial x_i}, n \right\rangle$, we define for $x \in \Gamma$ the single layer operator V, double layer operator K, adjoint double layer operator $K^\top$ and hypersingular integral operator W as

$$
V\mu(x) = \int_\Gamma G(x,y)\mu(y) ds_y, \qquad K v(x) = \int_\Gamma \left(\mathbb{T}_y G(x,y) \right)^\top v(y) ds_y \,, \tag{4}
$$

$$
K^\top \mu(x) = \mathbb{T}_x \int_\Gamma G(x,y)\mu(y) ds_y, \qquad W v(x) = -\mathbb{T}_x \int_\Gamma \left(\mathbb{T}_y G(x,y) \right)^\top v(y) ds_y \,, \tag{5}
$$

see [12] for transmission problems in linear elasticity and [16] for contact problems in linear elastostatics. The Poincaré-Steklov operator $S := W + (K + \frac{1}{2})^\top V^{-1}(K + \frac{1}{2})$ is a Dirichlet-to-Neumann mapping [8]:

$$
\langle S u, v \rangle = \langle \sigma(u)n, v \rangle = (\sigma(u_\Omega), \epsilon(v_\Omega))_{0,\Omega} \;.
$$

It is $H^{\frac{1}{2}}(\Gamma)$-continuous and $\widetilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)$-coercive, where $\widetilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)$ denotes the closed subspace of $H^{\frac{1}{2}}(\Gamma)$ of functions supported in $\overline{\Gamma}_\Sigma$. Hence the (domain) variational inequality immediately yields the (boundary) variational inequality formulation: Find $u \in K$ with $K := \left\{ v \in \widetilde{H}^{1/2}(\Gamma_\Sigma) : v_n \leq g \text{ a.e. on } \Gamma_C \right\}$ such that

$$
\langle S u, v - u \rangle_{\Gamma_\Sigma} + j(v) - j(u) \geq \langle f, v - u \rangle_{\Gamma_N} \quad \forall v \in K \,. \tag{6}
$$

It is well known, e.g. [9, Theorems 3.13 and 3.14], [10] that there exists a unique solution to (6). Since neither K is trivial to discretize nor is the non-differentiable friction functional $j(v)$ easy to handle [15] it may be favorable to use an equivalent mixed formulation. Since $\overline{\Gamma}_D \cap \overline{\Gamma}_C = \emptyset$ by assumption, let

$$M^+(\mathcal{F}) := \left\{\mu \in \widetilde{H}^{-1/2}(\Gamma_C) : \langle \mu, v\rangle_{\Gamma_C} \leq \langle \mathcal{F}, |v_t|\rangle_{\Gamma_C} \; \forall v \in \widetilde{H}^{1/2}(\Gamma_\Sigma), v_n \leq 0\right\} \tag{7}$$

be the set of admissible Lagrange multipliers, in which the representative $\lambda = -\sigma(u)n$ is sought. Then, the mixed method is to find the pair $(u, \lambda) \in \widetilde{H}^{1/2}(\Gamma_\Sigma) \times M^+(\mathcal{F})$ such that (see [3])

$$\langle S u, v\rangle_{\Gamma_\Sigma} + \langle \lambda, v\rangle_{\Gamma_C} = \langle f, v\rangle_{\Gamma_N} \qquad \forall v \in \widetilde{H}^{1/2}(\Gamma_\Sigma), \tag{8a}$$

$$\langle u, \mu - \lambda\rangle_{\Gamma_C} \leq \langle g, \mu_n - \lambda_n\rangle_{\Gamma_C} \quad \forall \mu \in M^+(\mathcal{F}). \tag{8b}$$

Theorem 1. *For the mixed problem* (8) *the following hold:*

1. *The inf-sup condition is satisfied with a constant* $\widetilde{\beta} > 0$, *i.e.*

$$\widetilde{\beta}\, \|\mu\|_{\widetilde{H}^{-1/2}(\Gamma_C)} \leq \sup_{v \in \widetilde{H}^{1/2}(\Gamma_\Sigma)\setminus\{0\}} \frac{\langle \mu, v\rangle_{\Gamma_C}}{\|v\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}} \qquad \forall \mu \in \widetilde{H}^{-1/2}(\Gamma_C)\,. \tag{9}$$

2. *Any solution of* (8) *is also a solution of* (6).
3. *For the solution* $u \in K$ *of* (6) *there exists a* $\lambda \in M^+(\mathcal{F})$ *such that* (u, λ) *is a solution of* (8)
4. *There exists a unique solution to* (8)

Proof. *1. The inf-sup condition has been proven in [9, Theorem 3.2.1].*
2. and 3. follow as in [32, Section 3] with $\langle S u, v\rangle_{\Gamma_\Sigma} = (\sigma(u_\Omega), \epsilon(v_\Omega))_{0,\Omega}$ *for volume force* $f_\Omega \equiv 0$.
4. follows from the equivalence results 2. and 3., the inf-sup condition 1. and from the unique existence of the solution of (6) *proven in [9, Theorems 3.13 and 3.14].*

3. Stabilized mixed *hp*-boundary element discretization including Lagrange multiplier

Let $\mathcal{T}_{h,\Gamma}$ be a subdivison of Γ into straight line segments such that the endpoints of the boundary parts coincide with a node from that mesh. Furthermore, let $\mathcal{T}_h = \mathcal{T}_{h,\Gamma}|_{\Gamma_\Sigma}$, h the distribution of side lengths, p the polynomial degree on $\mathcal{T}_{h,\Gamma}$ which on each element specifies the polynomial degree on the reference interval and Θ_E the affine mapping mapping from $[-1, 1]$ onto $E \in \mathcal{T}_h$. Moreover, assume the mesh and polynomial degree distribution to be locally quasi-uniform. We consider the ansatz spaces

$$\mathcal{V}_{hp} := \left\{v^{hp} \in \widetilde{H}^{1/2}(\Gamma_\Sigma) : v^{hp}|_E \circ \Theta_E \in \left[\mathbb{P}_{p_E}([-1, 1])\right]^2 \forall E \in \mathcal{T}_h\right\} \subset C^0(\Gamma_\Sigma)\,, \tag{10}$$

$$\mathcal{V}^D_{hp} := \left\{\phi^{hp} \in H^{-1/2}(\Gamma) : \phi^{hp}|_E \circ \Theta_E \in \left[\mathbb{P}_{p_E-1}([-1, 1])\right]^2 \forall E \in \mathcal{T}_{h,\Gamma}\right\}\,. \tag{11}$$

Note that $v^{hp} = 0$ in the endpoints of Γ_Σ if $v^{hp} \in \mathcal{V}_{hp}$. The displacement field u^{hp} is sought in $\mathcal{V}_{hp}$. Let $i_{hp} : \mathcal{V}^D_{hp} \mapsto H^{-1/2}(\Gamma)$ be the canonical embedding and i^*_{hp} its dual. The space $\mathcal{V}^D_{hp}$ is used to construct the standard approximation [8] $S_{hp} := W + \left(K^\top + \frac{1}{2}\right) i_{hp} V^{-1}_{hp} i^*_{hp} \left(K + \frac{1}{2}\right)$ of S, where V_{hp} is the Galerkin realization of the single layer potential over $\mathcal{V}^D_{hp}$. For the discrete Lagrange multiplier let $\widehat{\mathcal{T}}_k$ be an additional subdivision of Γ_C. The discrete Lagrange multiplier is sought in

$$M^+_{kq}(\mathcal{F}) := \left\{\mu^{kq} \in L^2(\Gamma_C) : \mu^{kq}|_E(x) = \sum_{i=0}^{q_E} \mu^E_i B_{i,q_E}(\Psi^{-1}_E(x)) \; \forall E \in \widehat{\mathcal{T}}_k,\; (\mu^E_i)_n \geq 0,\; \left|(\mu^E_i)_t\right| \leq \mathcal{F}(\Psi_E(iq^{-1}_E))\right\}\,, \tag{12}$$

where B_{i,q_E} is the i-th Bernstein polynomial of degree q_E and Ψ_E is the affine mapping from $[0, 1]$ onto $E \in \widehat{\mathcal{T}}_k$. Since the Bernstein polynomials are non-negative and form a partition of unity, it is straight forward to show that $M^+_{kq}(\mathcal{F})$ is conforming, i.e. $M^+_{kq}(\mathcal{F}) \subset M^+(\mathcal{F})$, if $\mathcal{F}$ is linear. Since $M^+_{kq}(\mathcal{F})$ is chosen independently of $\mathcal{V}_{hp}$ it cannot be expected that the discrete inf-sup condition holds uniformly, i.e. independently of h, k, p and q, especially not for $\widehat{\mathcal{T}}_k = \mathcal{T}_h|_{\Gamma_C}$. To circumvent the need to restrict the set $M^+_{kq}(\mathcal{F})$, the discrete mixed formulation is stabilized analogously to [4] for FEM. That is, find the pair $(u^{hp}, \lambda^{kq}) \in \mathcal{V}_{hp} \times M^+_{kq}(\mathcal{F})$ such that

$$\left\langle S_{hp}u^{hp}, v^{hp}\right\rangle_{\Gamma_\Sigma} + \left\langle \lambda^{kq}, v^{hp}\right\rangle_{\Gamma_C} - \left\langle \gamma\left(\lambda^{kq} + S_{hp}u^{hp}\right), S_{hp}v^{hp}\right\rangle_{\Gamma_C} = \left\langle f, v^{hp}\right\rangle_{\Gamma_N} \qquad \forall v^{hp} \in \mathcal{V}_{hp}, \tag{13a}$$

$$\left\langle \mu^{kq} - \lambda^{kq}, u^{hp}\right\rangle_{\Gamma_C} - \left\langle \gamma\left(\mu^{kq} - \lambda^{kq}\right), \lambda^{kq} + S_{hp}u^{hp}\right\rangle_{\Gamma_C} \leq \left\langle g, \mu^{kq}_n - \lambda^{kq}_n\right\rangle_{\Gamma_C} \quad \forall \mu^{kq} \in M^+_{kq}(\mathcal{F})\,. \tag{13b}$$

Here, γ is a piecewise constant function on Γ_C such that $\gamma|_E = \gamma_0 h_E^{1+\beta} p_E^{-2-\eta}$ with constants $\gamma_0 > 0$, β, $\eta \geq 0$ for all elements $E \in \mathcal{T}_h|_{\Gamma_C}$. For the forthcoming analysis, γ must scale at least like hp^{-2} to be able to compensate the scaling factors of the polynomial inverse estimates. In the following we assume $h \leq 1$ and $p \geq 1$.
Alternatively, $M^+(\mathcal{F})$ is discretized such that the constraints are only satisfied in a discrete set of points, namely

$$\widetilde{M}^+_{kq}(\mathcal{F}) := \left\{\mu^{kq} \in L^2(\Gamma_C) : \mu^{kq}|_E \circ \Psi_E \in \left[\mathbb{P}_{q_E}([-1,1])\right]^2,\ \mu^{kq}_n(x) \geq 0,\ -\mathcal{F}(x) \leq \mu^{kq}_t(x) \leq \mathcal{F}(x) \text{ for } x \in G_{kq}\right\}\,, \tag{14}$$

where G_{kq} is a set of discrete points on Γ_C, e.g. affinely transformed Gauss-Legendre points (which are used in the following)), and μ^{kq} are linear combinations of Gauss-Legendre-Lagrange basis functions. Enforcing the constraints of the primal variable u in such a finite set of points has only been applied successfully in e.g. [15, 25, 27].

We point out that $M^+_{k1}(\mathcal{F}) = \widetilde{M}^+_{k1}(\mathcal{F})$ for $q = 1$ if in $\widetilde{M}^+_{k1}(\mathcal{F})$ the set of Gauss-Lobatto points are used instead of the set of Gauss-Legendre points. This is no longer true for higher order polynomials. Unless specifically stated otherwise, the proven results are true for both discretizations $M^+_{kq}(\mathcal{F})$ and $\widetilde{M}^+_{kq}(\mathcal{F})$.
We collect some results on S_{hp} which allow to prove existence and uniqueness of the solution of the mixed formulation (13). Here and in the following, in estimates we also write $h = \max\ h_E$ and $p = \min\ p_E$ for the maximal side length, resp. minimal polynomial degree in the discretisation.

Lemma 2 (Lemma 15 in [27]). *There holds:*

1. *S_{hp} is continuous from $\widetilde{H}^{1/2}(\Gamma_\Sigma)$ into $H^{-1/2}(\Gamma)$ and coercive on $\widetilde{H}^{1/2}(\Gamma_\Sigma) \times \widetilde{H}^{1/2}(\Gamma_\Sigma)$. We denote the operator norm by C_S and the coercivity constant by α_S.*
2. *$E_{hp} := S - S_{hp}$ is bounded from $\widetilde{H}^{1/2}(\Gamma_\Sigma)$ into $H^{-1/2}(\Gamma)$, and there exists constants C_E, $C > 0$ such that for $v \in \widetilde{H}^{1/2}(\Gamma_\Sigma)$*

$$\left\|E_{hp}v\right\|_{H^{-1/2}(\Gamma)} \leq C_E \left\|v\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \quad \text{and} \quad \left\|E_{hp}v\right\|_{H^{-1/2}(\Gamma)} \leq C \inf_{\phi_{hp}\in\mathcal{V}^D_{hp}} \left\|V^{-1}(K + \frac{1}{2})v - \phi_{hp}\right\|_{H^{-1/2}(\Gamma)}\,.$$

For the coercivity in part 1., we refer to [8]. The proof for the h–method there extends verbatim to hp, provided h is replaced by $\min\{h, p^{-1}\}$ and $\min\{h, p^{-1}\}$ is sufficiently small. Reference [2] bootstraps the result from [8] from small h to arbitrary h; again the proof for hp only replaces h by $\min\{h, p^{-1}\}$.

Lemma 3. *Let $\delta \in [0, \frac{1}{2}]$. There exists a constant $C(\delta) > 0$ independent of h and p such that*

$$\|E_{hp}v_{hp}\|_{H^{-\delta}(\Gamma)} \leq C(\delta)\frac{p^{1-2\delta}}{h^{1/2-\delta}}\|v_{hp}\|_{H^{1/2}(\Gamma)} \quad \forall v_{hp} \in V_{hp}\,.$$

Proof. *As $V : H^{-\delta}(\Gamma) \mapsto H^{1-\delta}(\Gamma)$ is bijective, we may set $z = V^{-1}(K + \frac{1}{2})v_{hp} \in H^{-\delta}(\Gamma)$ and estimate*

$$\|z\|_{H^{-\delta}(\Gamma)} \leq C \left\|(K + \frac{1}{2})v_{hp}\right\|_{H^{1-\delta}(\Gamma)} \leq C \left\|v_{hp}\right\|_{H^{1-\delta}(\Gamma)} \leq C\frac{p^{1-2\delta}}{h^{1/2-\delta}} \left\|v_{hp}\right\|_{H^{1/2}(\Gamma)}\,.$$

Here the last two inequalities follow from the mapping properties of the boundary integral operators [11] and the inverse polynomial estimate [13] by complex interpolation. Let $z_{hp} \in \mathcal{V}^D_{hp}$ be the unique solution of

$$\left\langle Vz_{hp}, \phi_{hp}\right\rangle_\Gamma = \left\langle (K + \frac{1}{2})v_{hp}, \phi_{hp}\right\rangle_\Gamma \quad \forall \phi_{hp} \in \mathcal{V}^D_{hp}\,.$$

From the $H^{-1/2}(\Gamma)$-coercivity of V it follows that

$$\|z_{hp}\|_{H^{-1/2}(\Gamma)} \leq C \left\|(K + \frac{1}{2})v_{hp}\right\|_{H^{1/2}(\Gamma)} \leq C \left\|v_{hp}\right\|_{H^{1/2}(\Gamma)}.$$

Thus we obtain

$$\begin{aligned}
\|E_{hp}v_{hp}\|_{H^{-\delta}(\Gamma)} &= \sup_{\tau\in H^{\delta}(\Gamma)\setminus\{0\}} \frac{\left\langle E_{hp}v_{hp}, \tau\right\rangle}{\|\tau\|_{H^{\delta}(\Gamma)}} = \sup_{\tau\in H^{\delta}(\Gamma)\setminus\{0\}} \frac{\left\langle (V^{-1} - i_{hp}V_{hp}^{-1}i_{hp}^{*})(K + \frac{1}{2})v_{hp}, (K + \frac{1}{2})\tau\right\rangle}{\|\tau\|_{H^{\delta}(\Gamma)}} \\
&= \sup_{\tau\in H^{\delta}\setminus\{0\}} \frac{\left\langle z - i_{hp}z_{hp}, (K + \frac{1}{2})\tau\right\rangle}{\|\tau\|_{H^{\delta}(\Gamma)}} \leq C \left\|z - i_{hp}z_{hp}\right\|_{H^{-\delta}(\Gamma)} \leq C\left(\|z\|_{H^{-\delta}(\Gamma)} + \left\|z_{hp}\right\|_{H^{-\delta}(\Gamma)}\right) \\
&\leq C\left(\|z\|_{H^{-\delta}(\Gamma)} + C\frac{p^{1-2\delta}}{h^{1/2-\delta}}\left\|z_{hp}\right\|_{H^{-1/2}(\Gamma)}\right) \leq C\frac{p^{1-2\delta}}{h^{1/2-\delta}}\left\|v_{hp}\right\|_{H^{1/2}(\Gamma)},
\end{aligned}$$

where we use the inequalities from above and the inverse polynomial estimate.

Lemma 4 (Lemma 3.2.7 in [9], Prop. 5.1 in [7]). *With $u \in \widetilde{H}^{1/2}(\Gamma_\Sigma)$, $u^{hp} \in \mathcal{V}_{hp}$, let*

$$\psi = V^{-1}(K + \frac{1}{2})u, \qquad \psi^{*}_{hp} = V^{-1}(K + \frac{1}{2})u^{hp}, \qquad \psi^{hp} = i_{hp}V_{hp}^{-1}i_{hp}^{*}(K + \frac{1}{2})u^{hp}. \tag{15}$$

Then there holds

$$\left\langle V(\psi^{*}_{hp} - \psi^{hp}), \phi^{hp}\right\rangle_{\Gamma} = 0 \quad \forall\phi^{hp} \in \mathcal{V}_{hp}^{D}$$

and

$$\|u - u^{hp}\|_W^2 + \|\psi - \psi^{hp}\|_V^2 = \left\langle S u - S_{hp}u^{hp}, u - u^{hp}\right\rangle_{\Gamma_\Sigma} + \left\langle V(\psi^{*}_{hp} - \psi^{hp}), \psi - \psi^{hp}\right\rangle_{\Gamma},$$

where $\|u - u^{hp}\|_W^2 = \langle W(u - u^{hp}), u - u^{hp}\rangle_\Gamma$ and $\|\psi - \psi^{hp}\|_V^2 = \langle V(\psi - \psi^{hp}), \psi - \psi^{hp}\rangle_\Gamma$.

Theorem 5. *Let $\mathcal{T}_h$ be a locally quasi-uniform mesh. Then there holds*

$$\sum_{E\in\mathcal{T}_h} \left\|\frac{h_E^{1/2}}{p_E} S_{hp}v^{hp}\right\|_{L^2(E)}^2 \leq C^2 \left\|v^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}^2 \quad \forall v^{hp} \in \mathcal{V}_{hp}. \tag{16}$$

Proof. *From the definition of S_{hp} follows that $S_{hp}v^{hp} = Wv^{hp} + (K^\top + \frac{1}{2})i_{hp}\eta^{hp}$ with $\eta^{hp} = V_{hp}^{-1}i_{hp}^{*}(K + \frac{1}{2})v^{hp} \in \mathcal{V}_{hp}^{D}$. In [1, Corollary 3.2] it is shown that*

$$\sum_{E\in\mathcal{T}_h} \left\|\frac{h_E^{1/2}}{p_E} Wv^{hp}\right\|_{L^2(E)}^2 \leq C^2 \left\|v^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}^2, \qquad \sum_{E\in\mathcal{T}_h} \left\|\frac{h_E^{1/2}}{p_E} K^\top\eta^{hp}\right\|_{L^2(E)}^2 \leq C^2 \left\|\eta^{hp}\right\|_{\widetilde{H}^{-1/2}(\Gamma_\Sigma)}^2$$

for the boundary integral operators associated to the Laplacian. For the integral operators (4), (5) *of the Lamé equation this can be done analogously. The assertion follows with the mapping properties of $V_{hp}^{-1}i_{hp}^{*}(K + \frac{1}{2})$ and the standard inverse polynomial inequality for the identity term, see e.g. [13].*

Lemma 6 (Coercivity). *There exist constants α_S and $C > 0$ independent of h, p, k and q such that*

$$\left\langle S_{hp}v^{hp}, v^{hp}\right\rangle_{\Gamma_\Sigma} - \left\langle \gamma S_{hp}v^{hp}, S_{hp}v^{hp}\right\rangle_{\Gamma_C} \geq (\alpha_S - \gamma_0 C)\left\|v^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}^2 \qquad \forall v \in \mathcal{V}_{hp} \tag{17}$$

for γ_0 sufficiently small, with $\alpha_{stab} := \alpha_S - \gamma_0 C > 0$.

Proof. *From Theorem 5, $h \le 1$, $p \ge 1$, β, $\eta \ge 0$ it follows that*

$$\left\langle \gamma S_{hp} v^{hp}, S_{hp} v^{hp} \right\rangle_{\Gamma_C} \le \gamma_0 \frac{h^\beta}{p^\eta} C \left\| v^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \le \gamma_0 C \left\| v^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}, \tag{18}$$

with $C > 0$ independent of of h, p, k and q. Hence, from the coercivity of S_{hp} there holds

$$\left\langle S_{hp} v^{hp}, v^{hp} \right\rangle_{\Gamma_\Sigma} - \left\langle \gamma S_{hp} v^{hp}, S_{hp} v^{hp} \right\rangle_{\Gamma_C} \ge (\alpha_S - \gamma_0 C) \left\| v^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}. \tag{19}$$

Theorem 7 (Existence / Uniqueness). *For γ_0 sufficiently small, the discrete, stabilized problem* (13) *has a unique solution.*

Proof. *In the standard manner it can be shown that* (13) *is equivalent to the saddle-point problem: Find $(u^{hp}, \lambda^{kq}) \in \mathcal{V}_{hp} \times M^+_{kq}(\mathcal{F})$ such that*

$$\mathcal{L}_\gamma(u^{hp}, \mu^{kq}) \le \mathcal{L}_\gamma(u^{hp}, \lambda^{kq}) \le \mathcal{L}_\gamma(v^{hp}, \lambda^{kq}) \qquad \forall v^{hp} \in \mathcal{V}_{hp},\ \forall \mu^{kq} \in M^+_{kq}(\mathcal{F}), \tag{20}$$

with $L(v^{hp}) = \langle f, v^{hp} \rangle_{\Gamma_N}$, $\widetilde{L}(\mu^{kq}) = \langle g, \mu^{kq} \rangle_{\Gamma_C}$ and

$$\mathcal{L}_\gamma(v^{hp}, \mu^{kq}) = \frac{1}{2} \langle S_{hp} v^{hp}, v^{hp} \rangle_{\Gamma_\Sigma} - L(v^{hp}) - \widetilde{L}(\mu^{kq}) + \left\langle \mu^{kq}, v^{hp} \right\rangle_{\Gamma_C} - \frac{1}{2} \left\langle \gamma(\mu^{kq} + S_{hp} v^{hp}), \mu^{kq} + S_{hp} v^{hp} \right\rangle_{\Gamma_C}. \tag{21}$$

Due to $\alpha_{stab} = \alpha_S - \gamma_0 C > 0$ for γ_0 sufficiently small,

$$\mathcal{L}_\gamma(v^{hp}, 0) = \frac{1}{2} \langle S_{hp} v^{hp}, v^{hp} \rangle_{\Gamma_\Sigma} - L(v^{hp}) - \frac{1}{2} \int_{\Gamma_C} \gamma (S_{hp} v^{hp})^2 ds \ge \frac{\alpha_{stab}}{2} \left\| v^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} - \|f\|_{H^{-1/2}(\Gamma_N)} \left\| v^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)},$$

and $\mathcal{L}_\gamma(0, \mu^{kq}) = -\frac{1}{2} \int_{\Gamma_C} \gamma (\mu^{kq})^2 ds - \widetilde{L}(\mu^{kq})$, $\mathcal{L}_\gamma$ is strictly convex and coercive in v^{hp} and strictly concave and coercive in μ^{kq}. Since it is also continuous on $\mathcal{V}_{hp} \times M^+_{kq}(\mathcal{F})$, and $\mathcal{V}_{hp}$, $M^+_{kq}(\mathcal{F})$ are non-empty, closed, convex sets, standard arguments (e.g. [21]) provide the existence of a unique solution.

In the absence of stabilization, i.e. $\gamma_0 = 0$, $\mathcal{L}_\gamma$ is only linear and thus not strictly concave in μ^{kq}. Here however, strict concavity is needed to avoid the use of the discrete inf-sup condition. Due to the conformity in the primal variable there trivially holds the following Galerkin orthogonality.

Lemma 8. *Let (u, λ), (u^{hp}, λ^{kq}) be the solution of* (8), (13) *respectively. Then there holds*

$$\langle S u - S_{hp} u^{hp}, v^{hp} \rangle_{\Gamma_\Sigma} + \left\langle \lambda - \lambda^{kq}, v^{hp} \right\rangle_{\Gamma_C} + \left\langle \gamma(\lambda^{kq} + S_{hp} u^{hp}), S_{hp} v^{hp} \right\rangle_{\Gamma_C} = 0 \quad \forall v^{hp} \in \mathcal{V}_{hp}.$$

The next result will be used in our error analysis in Section 4.

Lemma 9 (Stability). *For the solutions (u, λ) of* (8) *and (u^{hp}, λ^{kq}) of* (13)*, there exists a constant $\widetilde{C} > 0$, independent of h, p, k and q, such that*

$$(\alpha_S - \gamma_0 C) \left\| u^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left\| \gamma^{1/2} \lambda^{kq} \right\|^2_{L^2(\Gamma_C)} \le \left(\widetilde{C} \|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|\lambda\|_{\widetilde{H}^{-1/2}(\Gamma_C)} \right) \left\| u^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|g\|_{H^{1/2}(\Gamma_C)} \left\| \lambda^{kq}_n \right\|_{\widetilde{H}^{-1/2}(\Gamma_C)}. \tag{22}$$

Proof. *Choosing $\mu^{kq}_n = 0$ and $\mu^{kq}_n = 2\lambda^{kq}_n$, each with $\mu^{kq}_t = \lambda^{kq}_t$, in* (13b) *yields*

$$\left\langle \lambda^{kq}_n, u^{hp}_n \right\rangle_{\Gamma_C} - \left\langle \gamma \lambda^{kq}_n, \lambda^{kq}_n + (S_{hp} u^{hp}) n \right\rangle_{\Gamma_C} = \left\langle g, \lambda^{kq}_n \right\rangle_{\Gamma_C},$$

whereas $\mu^{kq}_n = \lambda^{kq}_n$ and $\mu^{kq}_t = 0$ yields

$$\left\langle -\lambda^{kq}_t, u^{hp}_t \right\rangle_{\Gamma_C} + \left\langle \gamma \lambda^{kq}_t, \lambda^{kq}_t + (S_{hp} u^{hp}) t \right\rangle_{\Gamma_C} \le 0.$$

Hence, (13a) *yields with $v^{hp} = u^{hp}$ and Lemma 6*

$$\begin{aligned}\left\langle f, u^{hp}\right\rangle_{\Gamma_N} &= \left\langle S_{hp}u^{hp}, u^{hp}\right\rangle_{\Gamma_\Sigma} - \left\langle \gamma S_{hp}u^{hp}, S_{hp}u^{hp}\right\rangle_{\Gamma_C} + \left\langle \lambda^{kq}, u^{hp}\right\rangle_{\Gamma_C} - \left\langle \gamma\lambda^{kq}, S_{hp}u^{hp}\right\rangle_{\Gamma_C}\\ &\geq \left\langle S_{hp}u^{hp}, u^{hp}\right\rangle_{\Gamma_\Sigma} - \left\langle \gamma S_{hp}u^{hp}, S_{hp}u^{hp}\right\rangle_{\Gamma_C} + \left\langle \gamma\lambda^{kq}, \lambda^{kq}\right\rangle_{\Gamma_C} + \left\langle g, \lambda_n^{kq}\right\rangle_{\Gamma_C}\\ &\geq (\alpha_S - \gamma_0 C)\left\|u^{hp}\right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left\|\gamma^{1/2}\lambda^{kq}\right\|^2_{L^2(\Gamma_C)} + \left\langle g, \lambda_n^{kq}\right\rangle_{\Gamma_C}\ .\end{aligned}$$

On the other hand from (8a) *with $v = u^{hp} \in \mathcal{V}_{hp} \subset \widetilde{H}^{1/2}(\Gamma_\Sigma)$ it follows that*

$$\left\langle f, u^{hp}\right\rangle_{\Gamma_N} \leq \left(\widetilde{C}\,\|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|\lambda\|_{\widetilde{H}^{-1/2}(\Gamma_C)}\right)\left\|u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)},$$

which completes the proof.

The above proof also shows the following sharpened estimates:

Corollary 10. *Let $\epsilon > 0$ be an arbitrary constant. If $C^0(\Gamma_C) \ni g \geq 0$ and $\lambda^{kq} \in \widetilde{M}^+_{kq}(\mathcal{F})$, then there exists a constant $\widetilde{C} > 0$, independent of h, p, k, q and ϵ, such that*

$$(\alpha_S - \gamma_0 C - \epsilon)\left\|u^{hp}\right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + (1-\epsilon)\left\|\gamma^{1/2}\lambda^{kq}\right\|^2_{L^2(\Gamma_C)} \leq \frac{1}{4\epsilon}\left(\widetilde{C}\,\|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|\lambda\|_{\widetilde{H}^{-1/2}(\Gamma_C)}\right)^2 + \frac{1}{4\epsilon}\left\|\gamma^{-1/2}\left(g - \mathcal{I}_{kq}g\right)\right\|^2_{L^2(\Gamma_C)}.$$

Proof. *Recall that the (affinely transformed) Gauss-Legendre quadrature with $q+1$ points integrates polynomials of degree $2q+1$ over the element E exactly and that it has positive weights $\omega_i^{(q+1,E)}$. Let $\mathcal{I}_{kq}$ be the interpolation operator in the Gauss-Legendre points G_{kq}, then $\lambda_n^{kq}\mathcal{I}_{kq}g \geq 0$ in G_{kq} and, thus,*

$$\begin{aligned}\left\langle g, \lambda_n^{kq}\right\rangle_{\Gamma_C} &= \left\langle g - \mathcal{I}_{kq}g, \lambda_n^{kq}\right\rangle_{\Gamma_C} + \left\langle \mathcal{I}_{kq}g, \lambda_n^{kq}\right\rangle_{\Gamma_C} = \left\langle g - \mathcal{I}_{kq}g, \lambda_n^{kq}\right\rangle_{\Gamma_C} + \sum_{E\in\widehat{\mathcal{T}}_k}\sum_{i=0}^{q_E+1}\underbrace{\omega_i^{(q_E+1,E)}\lambda_n^{kq}(x_i^{(q_E+1,E)})\left(\mathcal{I}_{kq}g\right)(x_i^{(q_E+1,E)})}_{\geq 0}\\ &\geq \left\langle g - \mathcal{I}_{kq}g, \lambda_n^{kq}\right\rangle_{\Gamma_C}\ .\end{aligned}$$

Hence,

$$\left(\widetilde{C}\,\|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|\lambda\|_{\widetilde{H}^{-1/2}(\Gamma_C)}\right)\left\|u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left|\left\langle g - \mathcal{I}_{kq}g, \lambda_n^{kq}\right\rangle_{\Gamma_C}\right| \geq (\alpha_S - \gamma_0 C)\left\|u^{hp}\right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left\|\gamma^{1/2}\lambda^{kq}\right\|^2_{L^2(\Gamma_C)}\ .$$

The assertion follows with

$$\left|\left\langle g - \mathcal{I}_{kq}g, \lambda_n^{kq}\right\rangle_{\Gamma_C}\right| \leq \left\|\gamma^{-1/2}\left(g - \mathcal{I}_{kq}g\right)\right\|_{L^2(\Gamma_C)}\left\|\gamma^{1/2}\lambda^{kq}\right\|_{L^2(\Gamma_C)} \leq \frac{1}{4\epsilon}\left\|\gamma^{-1/2}\left(g - \mathcal{I}_{kq}g\right)\right\|^2_{L^2(\Gamma_C)} + \epsilon\left\|\gamma^{1/2}\lambda^{kq}\right\|^2_{L^2(\Gamma_C)}$$

for arbitrary $\epsilon > 0$ and Young's inequality.

Corollary 11. *If $\lambda^{kq} \in M^+_{kq}(\mathcal{F})$, i.e. $\lambda_n^{kq} \geq 0$, and if $g \geq 0$, then there exists a constant $\widetilde{C} > 0$, independent of h, p, k and q, such that*

$$(\alpha_S - \gamma_0 C)\left\|u^{hp}\right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left\|\gamma^{1/2}\lambda^{kq}\right\|^2_{L^2(\Gamma_C)} \leq \left(\widetilde{C}\,\|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|\lambda\|_{\widetilde{H}^{-1/2}(\Gamma_C)}\right)\left\|u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}. \tag{23}$$

4. A priori error estimates

Lemma 12. *Let (u,λ), (u^{hp},λ^{kq}) be the solutions of* (8), (13), *respectively, and $\lambda \in L^2(\Gamma_C)$. There holds*

$$\left\|\gamma^{\frac{1}{2}}(\lambda - \lambda^{kq})\right\|^2_{L_2(\Gamma_C)} \leq -\left\langle \lambda - \lambda^{kq}, u^{hp} - u\right\rangle_{\Gamma_C} + R_\gamma(u,\lambda,u^{hp},\lambda^{kq},g;\mu,\mu^{kq}), \tag{24}$$

where for any $\mu \in L^2(\Gamma_C) \cap M^+(\mathcal{F})$, $\mu^{kq} \in M^+_{kq}(\mathcal{F})$ (or $\mu^{kq} \in \widetilde{M}^+_{kq}(\mathcal{F})$ depending on the selected discretization) we define

$$\begin{aligned} R_\gamma(u,\lambda,u^{hp},\lambda^{kq},g;\mu,\mu^{kq}) := &\left\langle \lambda^{kq}-\mu, u\right\rangle_{\Gamma_C} + \left\langle \lambda-\mu^{kq}, u^{hp}-\gamma(\lambda^{kq}+Su^{hp})\right\rangle_{\Gamma_C} + \left\langle \gamma(\lambda-\lambda^{kq}), \lambda+Su^{hp}\right\rangle_{\Gamma_C} \\ &- \left\langle \gamma(\mu^{kq}-\lambda^{kq}), E_{hp}u^{hp}\right\rangle_{\Gamma_C} + \left\langle g, \mu^{kq}_n-\lambda^{kq}_n+\mu_n-\lambda_n\right\rangle_{\Gamma_C}. \end{aligned} \tag{25}$$

Proof. *First note that*

$$\left\|\gamma^{\frac{1}{2}}(\lambda-\lambda^{kq})\right\|^2_{L_2(\Gamma_C)} = \langle\gamma\lambda,\lambda\rangle_{\Gamma_C} - 2\left\langle\gamma\lambda,\lambda^{kq}\right\rangle_{\Gamma_C} + \left\langle\gamma\lambda^{kq},\lambda^{kq}\right\rangle_{\Gamma_C}.$$

Rearranging (13b) *we get*

$$\left\langle\gamma\lambda^{kq},\lambda^{kq}\right\rangle_{\Gamma_C} \le \left\langle\gamma\lambda^{kq},\mu^{kq}\right\rangle_{\Gamma_C} - \left\langle\mu^{kq}-\lambda^{kq},u^{hp}\right\rangle_{\Gamma_C} + \left\langle\gamma(\mu^{kq}-\lambda^{kq}),S_{hp}u^{hp}\right\rangle_{\Gamma_C} + \left\langle g,\mu^{kq}_n-\lambda^{kq}_n\right\rangle_{\Gamma_C} \quad \forall\mu^{kq}\in M^+_{kq}(\mathcal{F}).$$

Adding (8b) *results in*

$$\begin{aligned} \left\|\gamma^{\frac{1}{2}}(\lambda-\lambda^{kq})\right\|^2_{L_2(\Gamma_C)} \le\ &\langle\gamma\lambda,\lambda\rangle_{\Gamma_C} - 2\left\langle\gamma\lambda,\lambda^{kq}\right\rangle_{\Gamma_C} + \left\langle\gamma\lambda^{kq},\mu^{kq}\right\rangle_{\Gamma_C} - \left\langle\mu^{kq}-\lambda^{kq},u^{hp}\right\rangle_{\Gamma_C} + \left\langle\gamma(\mu^{kq}-\lambda^{kq}),S_{hp}u^{hp}\right\rangle_{\Gamma_C} \\ &+ \left\langle g,\mu^{kq}_n-\lambda^{kq}_n\right\rangle_{\Gamma_C} \\ \le &-\left\langle\lambda-\lambda^{kq},u^{hp}-u\right\rangle_{\Gamma_C} + \left\langle\lambda-\mu^{kq},u^{hp}\right\rangle_{\Gamma_C} + \left\langle\lambda^{kq}-\mu,u\right\rangle_{\Gamma_C} + \langle\gamma\lambda,\lambda\rangle_{\Gamma_C} - 2\left\langle\gamma\lambda,\lambda^{kq}\right\rangle_{\Gamma_C} \\ &+ \left\langle\gamma\lambda^{kq},\mu^{kq}\right\rangle_{\Gamma_C} + \left\langle\gamma(\mu^{kq}-\lambda^{kq}),S_{hp}u^{hp}\right\rangle_{\Gamma_C} + \left\langle g,\mu^{kq}_n-\lambda^{kq}_n+\mu_n-\lambda_n\right\rangle_{\Gamma_C}. \end{aligned}$$

Rearranging the terms and adding the zero $\left\langle\gamma\lambda,Su^{hp}-Su^{hp}\right\rangle_{\Gamma_C}$ gives (with $E_{hp}=S-S_{hp}$)

$$\begin{aligned} &\left\langle\lambda-\mu^{kq},u^{hp}\right\rangle_{\Gamma_C} + \langle\gamma\lambda,\lambda\rangle_{\Gamma_C} - 2\left\langle\gamma\lambda,\lambda^{kq}\right\rangle_{\Gamma_C} + \left\langle\gamma\lambda^{kq},\mu^{kq}\right\rangle_{\Gamma_C} + \left\langle\gamma(\mu^{kq}-\lambda^{kq}),S_{hp}u^{hp}\right\rangle_{\Gamma_C} \\ &= \left\langle\lambda-\mu^{kq},u^{hp}-\gamma(\lambda^{kq}+Su^{hp})\right\rangle_{\Gamma_C} + \left\langle\gamma(\lambda-\lambda^{kq}),\lambda+Su^{hp}\right\rangle_{\Gamma_C} - \left\langle\gamma(\mu^{kq}-\lambda^{kq}),E_{hp}u^{hp}\right\rangle_{\Gamma_C}, \end{aligned}$$

which yields the assertion.

Theorem 13. *Let (u,λ), (u^{hp},λ^{kq}) be the solutions of* (8), (13), *respectively, and ψ,ψ^{hp} as in* (15). *If $\lambda\in L^2(\Gamma_C)$, then there holds with arbitrary $v^{hp}\in\mathcal{V}_{hp}$, $\phi^{hp}\in\mathcal{V}^D_{hp}$, $\mu\in M^+(\mathcal{F})\cap L^2(\Gamma_C)$, $\mu^{kq}\in M^+_{kq}(\mathcal{F})$ (or $\mu^{kq}\in\widetilde{M}^+_{kq}(\mathcal{F})$ depending on the selected discretization)*

$$\begin{aligned} &(\alpha_W-2\epsilon_1)\|u-u^{hp}\|^2_{\widetilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)} + (\alpha_V-\epsilon_2)\|\psi-\psi^{hp}\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \|\gamma^{\frac{1}{2}}(\lambda-\lambda^{kq})\|^2_{L^2(\Gamma_C)} \le \left(\frac{(C+C_E)^2}{4\epsilon_1}+\frac{C_2}{2}\right)\|u-v^{hp}\|^2_{\widetilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)} \\ &+\left(\frac{(C_K+1/2)^2}{4\epsilon_1}+\frac{1}{4\epsilon_2}+\frac{C_2}{2}\right)\|\psi-\phi^{hp}\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \left\langle\lambda-\lambda^{kq},u-v^{hp}\right\rangle_{\Gamma_C} + \left\langle\lambda^{kq}-\mu,u\right\rangle_{\Gamma_C} \\ &+\left\langle\lambda-\mu^{kq},u^{hp}-\gamma(\lambda^{kq}+Su^{hp})\right\rangle_{\Gamma_C} - \left\langle\gamma(\mu^{kq}-\lambda^{kq}),E_{hp}u^{hp}\right\rangle_{\Gamma_C} + \left\langle g,\mu^{kq}_n-\lambda^{kq}_n+\mu_n-\lambda_n\right\rangle_{\Gamma_C} \\ &-\left\langle\gamma(\lambda^{kq}+S_{hp}u^{hp}),E_{hp}(u^{hp}-v^{hp})\right\rangle_{\Gamma_C} + \left\langle\gamma(\lambda-\lambda^{kq}),\lambda+Sv^{hp}\right\rangle_{\Gamma_C} + \left\langle\gamma(\lambda+Su^{hp}),S(u^{hp}-v^{hp})\right\rangle_{\Gamma_C} \\ &-\left\langle\gamma E_{hp}u^{hp},S(u^{hp}-v^{hp})\right\rangle_{\Gamma_C}, \end{aligned}$$

with constants $\alpha_W,\alpha_V,C,C_E,C_2,C_K>0$ independent of h, k, p and q and $\epsilon_1,\epsilon_2>0$ arbitrary.

Proof. *By the coercivity of W and V, and by Lemmas 4, 8 and 12 there holds for all $v^{hp}\in\mathcal{V}_{hp}$, $\mu^{kq}\in M^+_{kq}(\mathcal{F})$ or*

$\mu^{kq} \in \widetilde{M}^+_{kq}(\mathcal{F})$ depending on the selected discretization, $\mu \in L^2(\Gamma_C) \cap M^+(\mathcal{F})$, that

$$
\begin{aligned}
&\alpha_W \left\|u - u^{hp}\right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \alpha_V \left\|\psi - \psi^{hp}\right\|^2_{H^{-1/2}(\Gamma)} + \left\|\gamma^{1/2}(\lambda - \lambda^{kq})\right\|^2_{L^2(\Gamma_C)} \\
&\leq \left\langle S u - S_{hp} u^{hp}, u - u^{hp}\right\rangle_{\Gamma_\Sigma} + \left\langle V(\psi^*_{hp} - \psi^{hp}), \psi - \psi^{hp}\right\rangle_\Gamma + \left\|\gamma^{1/2}(\lambda - \lambda^{kq})\right\|^2_{L^2(\Gamma_C)} \\
&= \left\langle S u - S_{hp} u^{hp}, u - v^{hp}\right\rangle_{\Gamma_\Sigma} + \left\langle V(\psi^*_{hp} - \psi^{hp}), \psi - \psi^{hp}\right\rangle_\Gamma + \left\langle \lambda - \lambda^{kq}, u^{hp} - v^{hp}\right\rangle_{\Gamma_C} \\
&\quad + \left\langle \gamma(\lambda^{kq} + S_{hp} u^{hp}), S_{hp}(u^{hp} - v^{hp})\right\rangle_{\Gamma_C} + \left\|\gamma^{1/2}(\lambda - \lambda^{kq})\right\|^2_{L^2(\Gamma_C)} \\
&\leq \underbrace{\left\langle S u - S_{hp} u^{hp}, u - v^{hp}\right\rangle_{\Gamma_\Sigma}}_{=:A} + \underbrace{\left\langle V(\psi^*_{hp} - \psi^{hp}), \psi - \psi^{hp}\right\rangle_\Gamma}_{=:B} + \left\langle \lambda - \lambda^{kq}, u - v^{hp}\right\rangle_{\Gamma_C} + R_\gamma(u, \lambda, u^{hp}, \lambda^{kq}, g; \mu, \mu^{kq}) \\
&\quad + \left\langle \gamma(\lambda^{kq} + S_{hp} u^{hp}), S_{hp}(u^{hp} - v^{hp})\right\rangle_{\Gamma_C}.
\end{aligned}
$$

It remains to estimate A, B and the last two terms. Since, $S u - S_{hp} u^{hp} = S(u - u^{hp}) + E_{hp}(u^{hp} - u) + E_{hp} u$ we obtain with Lemma 2 and $\psi = V^{-1}(K + 1/2)u$ that

$$
A \leq \left[(C + C_E)\left\|u - u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + C_2 \left\|\psi - \phi^{hp}\right\|_{H^{-1/2}(\Gamma)}\right] \left\|u - v^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \quad \forall \phi^{hp} \in \mathcal{V}^D_{hp}.
$$

From Lemma 4, adding the zero $\left\langle V(\psi - \psi), \psi - \phi^{hp}\right\rangle_\Gamma$ and (15) *it follows that*

$$
\begin{aligned}
B &= \left\langle V(\psi^*_{hp} - \psi^{hp}), \psi - \phi^{hp}\right\rangle_\Gamma = \left\langle \left(K + \frac{1}{2}\right)(u^{hp} - u), \psi - \phi^{hp}\right\rangle_\Gamma + \left\langle V(\psi - \psi^{hp}), \psi - \phi^{hp}\right\rangle_\Gamma \\
&\leq \left[\left(C_K + \frac{1}{2}\right)\left\|u - u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left\|\psi - \psi^{hp}\right\|_{H^{-1/2}(\Gamma)}\right] \left\|\psi - \phi^{hp}\right\|_{H^{-1/2}(\Gamma)} \quad \forall \phi^{hp} \in \mathcal{V}^D_{hp}.
\end{aligned}
$$

Note that ($E_{hp} = S - S_{hp}$)

$$
\begin{aligned}
&\left\langle \gamma(\lambda^{kq} + S_{hp} u^{hp}), S_{hp}(u^{hp} - v^{hp})\right\rangle_{\Gamma_C} + \left\langle \gamma(\lambda - \lambda^{kq}), \lambda + S u^{hp}\right\rangle_{\Gamma_C} = -\left\langle \gamma(\lambda^{kq} + S_{hp} u^{hp}), E_{hp}(u^{hp} - v^{hp})\right\rangle_{\Gamma_C} \\
&\quad + \left\langle \gamma(\lambda^{kq} + S_{hp} u^{hp}), S(u^{hp} - v^{hp})\right\rangle_{\Gamma_C} + \left\langle \gamma(\lambda - \lambda^{kq}), \lambda + S v^{hp} + S(u^{hp} - v^{hp})\right\rangle_{\Gamma_C} \\
&= -\left\langle \gamma(\lambda^{kq} + S_{hp} u^{hp}), E_{hp}(u^{hp} - v^{hp})\right\rangle_{\Gamma_C} + \left\langle \gamma(\lambda - \lambda^{kq}), \lambda + S v^{hp}\right\rangle_{\Gamma_C} + \left\langle \gamma(\lambda + S_{hp} u^{hp}), S(u^{hp} - v^{hp})\right\rangle_{\Gamma_C} \\
&= -\left\langle \gamma(\lambda^{kq} + S_{hp} u^{hp}), E_{hp}(u^{hp} - v^{hp})\right\rangle_{\Gamma_C} + \left\langle \gamma(\lambda - \lambda^{kq}), \lambda + S v^{hp}\right\rangle_{\Gamma_C} + \left\langle \gamma(\lambda + S u^{hp}), S(u^{hp} - v^{hp})\right\rangle_{\Gamma_C} \\
&\quad - \left\langle \gamma E_{hp} u^{hp}, S(u^{hp} - v^{hp})\right\rangle_{\Gamma_C}.
\end{aligned}
$$

Application of Young's inequality yields the assertion.

Following [30], there holds by exploiting the exactness of the Gauss-Legendre quadrature:

Lemma 14. *If the Lagrange multiplier mesh is decomposable such that $\widehat{\mathcal{T}}_k = \mathcal{E}^*_k \cup \mathcal{E}^\pm_k$ with*

$$
\mathcal{E}^*_k := \left\{E \in \widehat{\mathcal{T}}_k : |\lambda_t| < \mathcal{F} \ \forall x \in E\right\}, \qquad \mathcal{E}^\pm_k := \left\{E \in \widehat{\mathcal{T}}_k : \pm\lambda_t \geq 0 \ \forall x \in E\right\}, \tag{26}
$$

$\lambda^{kq} \in \widetilde{M}^+_{kq}(\mathcal{F})$, $0 \leq g \in H^{1+\alpha}(\Gamma_C)$ and $u \in H^{1+\alpha}(\Gamma)$ with $\alpha \in [0, \frac{1}{2})$, then there exists a constant $C > 0$ independent of h, p, k and q such that

$$
\inf_{\mu \in M^+(\mathcal{F})} \left\langle \lambda^{kq}_n - \mu_n, u_n - g\right\rangle_{\Gamma_C} + \left\langle \lambda^{kq}_t - \mu_t, u_t\right\rangle_{\Gamma_C} \leq C \frac{k^{1+\alpha}}{q^{1+\alpha}} \left(\gamma_0^{-1/2} \frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}} + \gamma_0^{-1} \frac{p^{(2+\eta)}}{h^{(1+\beta)}} \frac{k^{1+\alpha}}{q^{1+\alpha}} + 1\right). \tag{27}
$$

Proof. *From Corollary10 it follows that*

$$\left\|\gamma^{1/2}\lambda^{kq}\right\|_{L^2(\Gamma_C)} \leq C\left(\|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|\lambda\|_{\widetilde{H}^{-1/2}(\Gamma_C)} + \gamma_0^{-1/2}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\|g\|_{H^{1+\alpha}(\Gamma_C)}\right). \tag{28}$$

Recall that the (affinely transformed) Gauss-Legendre quadrature with $q+1$ points integrates polynomials of degree $2q+1$ over the element E exactly and it has positive weights $\omega_i^{(q+1,E)}$. Since $u_n \leq g$ a.e. on Γ_C we obtain for the first of the two terms with $\mu_n = 0$ that

$$\begin{aligned}
\left\langle \lambda_n^{kq} - \mu_n, u_n - g\right\rangle_{\Gamma_C} &= \int_{\Gamma_C}\lambda_n^{kq}\left((u_n-g) - \mathcal{I}_{kq}(u_n-g)\right)\,ds + \int_{\Gamma_C}\lambda_n^{kq}\mathcal{I}_{kq}(u_n-g)\,ds\\
&= \int_{\Gamma_C}\lambda_n^{kq}\left((u_n-g) - \mathcal{I}_{kq}(u_n-g)\right)\,ds + \sum_{E\in\widehat{\mathcal{T}}_k}\sum_{i=0}^{q_E+1}\underbrace{\omega_i^{(q_E+1,E)}\lambda_n^{kq}(x_i^{(q_E+1,E)})\left(\mathcal{I}_{kq}(u_n-g)\right)(x_i^{(q_E+1,E)})}_{\leq 0}\\
&\leq \left\|\gamma^{1/2}\lambda^{kq}\right\|_{L^2(\Gamma_C)}\left(\left\|\gamma^{-1/2}\left(u_n - \mathcal{I}_{kq}u_n\right)\right\|_{L^2(\Gamma_C)} + \left\|\gamma^{-1/2}\left(g - \mathcal{I}_{kq}g\right)\right\|_{L^2(\Gamma_C)}\right)\\
&\leq C\gamma_0^{-1/2}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\left(\|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|\lambda\|_{\widetilde{H}^{-1/2}(\Gamma_C)} + \gamma_0^{-1/2}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\|g\|_{H^{1+\alpha}(\Gamma_C)}\right)\left(\|u\|_{H^{1+\alpha}(\Gamma)} + \|g\|_{H^{1+\alpha}(\Gamma_C)}\right)
\end{aligned}$$

with $\lambda^{kq} \in \widetilde{M}_{kq}^+(\mathcal{F})$, the exact integration with a quadrature formula adjusted to the constraints in $\widetilde{M}_{kq}^+(\mathcal{F})$ and with (28).
Given the decomposition $\widehat{\mathcal{T}}_k = \mathcal{E}_k^* \cup \mathcal{E}_k^\pm$ we choose $\mu_t|_E = \pm\mathcal{F}$ for $E \in \mathcal{E}_k^\pm$ and $\mu_t|_E = 0$ for $E \in \mathcal{E}_k^*$. From (2) *we deduce that $\pm u_t|_E \geq 0$ for $E \in \mathcal{E}_k^\pm$ and $u_t|_E = 0$ for $E \in \mathcal{E}_k^*$. Hence, for $\lambda^{kq} \in \widetilde{M}_{kq}^+(\mathcal{F})$ we obtain $(\lambda_t^{kq} - \mu_t)u_t \leq 0$ in the Gauss-Legendre points G_{kq}. Therefore*

$$\begin{aligned}
\left\langle \lambda_t^{kq} - \mu_t, u_t\right\rangle_{\Gamma_C} &= \int_{\Gamma_C}\left(\lambda_t^{kq} - \mu_t\right)\left(u_t - \mathcal{I}_{kq}u_t\right)\,ds + \int_{\Gamma_C}\left(\lambda_t^{kq} - \mu_t\right)\mathcal{I}_{kq}u_t\,ds\\
&= \int_{\Gamma_C}\left(\lambda_t^{kq} - \mu_t\right)\left(u_t - \mathcal{I}_{kq}u_t\right)\,ds + \sum_{E\in\widehat{\mathcal{T}}_k}\sum_{i=0}^{q_E+1}\underbrace{\omega_i^{(q_E+1,E)}\left(\lambda_t^{kq} - \mu_t\right)\circ(x_i^{(q_E+1,E)})\left(\mathcal{I}_{kq}u_t\right)\circ(x_i^{(q_E+1,E)})}_{\leq 0}\\
&\leq \left\|\gamma^{1/2}\lambda^{kq}\right\|_{L^2(\Gamma_C)}\left\|\gamma^{-1/2}\left(u_t - \mathcal{I}_{kq}u_t\right)\right\|_{L^2(\Gamma_C)} + \|\mu_t\|_{L^2(\Gamma_C)}\left\|u_t - \mathcal{I}_{kq}u_t\right\|_{L^2(\Gamma_C)}\\
&\leq \frac{C}{\gamma_0^{1/2}}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\left(\|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|\lambda\|_{\widetilde{H}^{-1/2}(\Gamma_C)} + \gamma_0^{-1/2}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\|g\|_{H^{1+\alpha}(\Gamma_C)}\right)\|u\|_{H^{1+\alpha}(\Gamma)}\\
&\quad + C\frac{k^{1+\alpha}}{q^{1+\alpha}}\|\mathcal{F}\|_{L^2(\Gamma_C)}\|u\|_{H^{1+\alpha}(\Gamma)}\,.
\end{aligned}$$

Remark 15.
1. *The condition $\widehat{\mathcal{T}}_k = \mathcal{E}_k^* \cup \mathcal{E}_k^\pm$ requires that the continuous Lagrange multiplier λ_t does not change its sign and that it does not take its upper or lower bound on the same element $E \in \widehat{\mathcal{T}}_k$.*
2. *If $\lambda_t \in C^0(\Gamma_C)$, $\widehat{\mathcal{T}}_k = \mathcal{E}_k^* \cup \mathcal{E}_k^\pm$, can always be achieved if the mesh size k is sufficiently small.*
3. *The condition $\widehat{\mathcal{T}}_k = \mathcal{E}_k^* \cup \mathcal{E}_k^\pm$ is also fulfilled if the "critical" points of discontinuity of λ_t coincide with nodes of the mesh $\widehat{\mathcal{T}}_k$.*

Theorem 16. *Let $(u,\lambda) \in H^{1+\alpha}(\Gamma) \times H^\alpha(\Gamma_C) \cap C^0(\Gamma_C)$ and $(u^{hp}, \lambda^{kq}) \in \mathcal{V}_{hp} \times \widetilde{M}_{kq}^+(\mathcal{F})$ be the solutions of* (8), (13), *respectively, with $0 \leq g \in H^{1+\alpha}(\Gamma_C)$ and $\alpha \in [0, \frac{1}{2})$. If $\widehat{\mathcal{T}}_k = \mathcal{E}_k^* \cup \mathcal{E}_k^\pm$, then there exists a constant $C > 0$ independent of h, p, k, q, $\beta \geq 0$ and $\eta \geq 0$ such that there holds*

$$\|u - u^{hp}\|^2_{\widetilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)} + \|\gamma^{\frac{1}{2}}(\lambda - \lambda^{kq})\|^2_{L^2(\Gamma_C)} + \|\psi - \psi^{hp}\|^2_{H^{-\frac{1}{2}}(\Gamma)} \leq C\left(\frac{h^\beta}{p^\eta} + \frac{h^{1+2\alpha-\beta}}{p^{2\alpha-\eta}} + \frac{k^\alpha}{q^\alpha} + \frac{k^{1+\alpha}}{q^{1+\alpha}}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}} + \frac{k^{2+2\alpha}}{q^{2+2\alpha}}\frac{p^{2+\eta}}{h^{1+\beta}}\right),$$

with ψ, ψ^{hp} given as in (15).

Proof. We apply Theorem 13 and estimate the individual terms. For that we use the following results:
Let $\mathcal{I}_{hp}$ be the interpolation operator in the Gauss-Lobatto points G_{hp} on Γ_Σ and $\mathcal{I}_{kq}$ be the interpolation operator in the Gauss-Legendre points G_{kq}. Since $u \in \widetilde{H}^{1/2}(\Gamma_\Sigma)$ and $\mathcal{I}_{hp}u$ can be extended continuously by zero onto the whole Γ (denoted $\mathcal{I}_{hp,0}u$) we have [6, Theorem 4.7 and Theorem 3.4]:

$$\left\|u - \mathcal{I}_{hp}u\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} = \left\|u - \mathcal{I}_{hp,0}u\right\|_{H^{1/2}(\Gamma)} \le C\frac{h^{1/2+\alpha}}{p^{1/2+\alpha}}\left\|u\right\|_{H^{1+\alpha}(\Gamma)},$$

$$\left\|u - \mathcal{I}_{hp,0}u\right\|_{H^s(\Gamma)} \le C\frac{h^{1-s+\alpha}}{p^{1-s+\alpha}}\left\|u\right\|_{H^{1+\alpha}(\Gamma)},\ s \in \{0,1\} \quad \text{and} \quad \left\|\lambda - \mathcal{I}_{kq}\lambda\right\|_{L^2(\Gamma_C)} \le C\frac{k^\alpha}{q^\alpha}\left\|\lambda\right\|_{H^\alpha(\Gamma_C)}.$$

In particular, $\mathcal{I}_{kq}\lambda \in \widetilde{M}^+_{kq}(\mathcal{F})$. We also note that from the mapping properties of the boundary integral operators and Lemma 3:

$$\|S v\|_{L^2(\Gamma)} \le C\,\|v\|_{H^1(\Gamma)} \quad \forall v \in H^1(\Gamma), \tag{29}$$

$$\left\|E_{hp}v^{hp}\right\|_{L^2(\Gamma)} \le C\frac{p}{h^{1/2}}\left\|v^{hp}\right\|_{H^{1/2}(\Gamma)} \quad \forall v \in \mathcal{V}_{hp}. \tag{30}$$

Furthermore, we need the polynomial inverse estimates, see e.g. [13] and using complex interpolation:

$$\left\|v^{hp}\right\|_{H^s(\Gamma)} \le C\frac{p^{2s}}{h^s}\left\|v^{hp}\right\|_{L^2(\Gamma)} \qquad \forall v^{hp} \in \mathcal{V}_{hp},\ s \ge 0, \tag{31}$$

$$\left\|v^{hp}\right\|_{H^1(\Gamma)} \le C\frac{p}{h^{1/2}}\left\|v^{hp}\right\|_{H^{1/2}(\Gamma)} \quad \forall v^{hp} \in \mathcal{V}_{hp}. \tag{32}$$

In the error estimate the following terms appear in several places.
From (29) and $v^{hp} = \mathcal{I}_{hp,0}u$ it follows that

$$\left\|\gamma^{1/2}S(u - v^{hp})\right\|_{L^2(\Gamma_C)} \le \gamma_0^{1/2}C\frac{h^{(1+\beta)/2}}{p^{(2+\eta)/2}}\left\|u - v^{hp}\right\|_{H^1(\Gamma)} \le \gamma_0^{1/2}C\frac{h^{(1+\beta)/2+\alpha}}{p^{(2+\eta)/2+\alpha}}\left\|u\right\|_{H^{1+\alpha}(\Gamma)}. \tag{33}$$

From (29), (32) and the triangle inequality it follows that

$$\begin{aligned}\left\|\gamma^{1/2}S(u^{hp} - v^{hp})\right\|_{L^2(\Gamma_C)} &\le \gamma_0^{1/2}C\frac{h^{(1+\beta)/2}}{p^{(2+\eta)/2}}\left\|u^{hp} - v^{hp}\right\|_{H^1(\Gamma)} \le \gamma_0^{1/2}C\frac{h^{\beta/2}}{p^{\eta/2}}\left\|u^{hp} - v^{hp}\right\|_{H^{1/2}(\Gamma)} = \gamma_0^{1/2}C\frac{h^{\beta/2}}{p^{\eta/2}}\left\|u^{hp} - v^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}\\ &\le \gamma_0^{1/2}C\frac{h^{\beta/2}}{p^{\eta/2}}\left\|u^{hp} - u\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \gamma_0^{1/2}C\frac{h^{\beta/2}}{p^{\eta/2}}\left\|u - v^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}\\ &\le \gamma_0^{1/2}C\frac{h^{\beta/2}}{p^{\eta/2}}\left\|u^{hp} - u\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \gamma_0^{1/2}C\frac{h^{(1+\beta)/2+\alpha}}{p^{(1+\eta)/2+\alpha}}\left\|u\right\|_{H^{1+\alpha}(\Gamma)}.\end{aligned} \tag{34}$$

Equation (30) and the triangle inequality now imply:

$$\left\|\gamma^{1/2}E_{hp}(u^{hp} - v^{hp})\right\|_{L^2(\Gamma_C)} \le \gamma_0^{1/2}C\frac{h^{\beta/2}}{p^{\eta/2}}\left\|u^{hp} - v^{hp}\right\|_{H^{1/2}(\Gamma)} \le \gamma_0^{1/2}C\frac{h^{\beta/2}}{p^{\eta/2}}\left\|u^{hp} - u\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \gamma_0^{1/2}C\frac{h^{(1+\beta)/2+\alpha}}{p^{(1+\eta)/2+\alpha}}\left\|u\right\|_{H^{1+\alpha}(\Gamma)}. \tag{35}$$

From (30) it follows that

$$\left\|\gamma^{1/2}E_{hp}u^{hp}\right\|_{L^2(\Gamma_C)} \le \gamma_0^{1/2}C\frac{h^{\beta/2}}{p^{\eta/2}}\left\|u^{hp}\right\|_{H^{1/2}(\Gamma)} = \gamma_0^{1/2}C\frac{h^{\beta/2}}{p^{\eta/2}}\left\|u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}. \tag{36}$$

With these approximation results at hand the remaining terms can be estimated. From the Cauchy-Schwarz inequality and Young's inequality follows

$$\begin{aligned}\langle\lambda - \lambda^{kq}, u - v^{hp}\rangle_{\Gamma_C} &\le \|\gamma^{1/2}(\lambda^{kq} - \lambda)\|_{L_2(\Gamma_C)}\|\gamma^{-1/2}(u - v^{hp})\|_{L_2(\Gamma_C)} \le \epsilon\|\gamma^{\frac{1}{2}}(\lambda^{kq} - \lambda)\|^2_{L^2(\Gamma_C)} + \frac{1}{4\epsilon\gamma_0}\frac{p^{2+\eta}}{h^{1+\beta}}\|u - v^{hp}\|^2_{L^2(\Gamma_C)}\\ &\le \epsilon\|\gamma^{\frac{1}{2}}(\lambda^{kq} - \lambda)\|^2_{L^2(\Gamma_C)} + \frac{C}{\epsilon\gamma_0}\frac{h^{1+2\alpha-\beta}}{p^{2\alpha-\eta}}\|u\|^2_{H^{1+\alpha}(\Gamma)} = \epsilon\|\gamma^{\frac{1}{2}}(\lambda^{kq} - \lambda)\|^2_{L^2(\Gamma_C)} + \frac{C}{\epsilon\gamma_0}\frac{h^{1+2\alpha-\beta}}{p^{2\alpha-\eta}}.\end{aligned} \tag{37}$$

From Lemma 14 and interpolation we obtain

$$
\begin{aligned}
&\inf_{\mu\in M^+(\mathcal{F})} \left\langle \lambda^{kq}-\mu, u\right\rangle_{\Gamma_C} + \left\langle g, \mu_n^{kq}-\lambda_n^{kq}+\mu_n-\lambda_n\right\rangle_{\Gamma_C} \\
&\quad = \inf_{\mu\in M^+(\mathcal{F})} \left\langle \lambda_n^{kq}-\mu_n, u_n-g\right\rangle_{\Gamma_C} + \left\langle \lambda_t^{kq}-\mu_t, u_t\right\rangle_{\Gamma_C} + C\frac{k^\alpha}{q^\alpha}\|\lambda\|_{H^\alpha(\Gamma_C)}\|g\|_{L^2(\Gamma_C)} \\
&\quad \le C\frac{k^{1+\alpha}}{q^{1+\alpha}}\left(\gamma_0^{-1/2}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}} + \gamma_0^{-1}\frac{p^{(2+\eta)}}{h^{(1+\beta)}}\frac{k^{1+\alpha}}{q^{1+\alpha}} + 1\right) + C\frac{k^\alpha}{q^\alpha}\|\lambda\|_{H^\alpha(\Gamma_C)}\|g\|_{L^2(\Gamma_C)} \\
&\quad \le \frac{C}{\gamma_0^{1/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}} + \frac{C}{\gamma_0}\frac{k^{2+2\alpha}}{q^{2+2\alpha}}\frac{p^{(2+\eta)}}{h^{(1+\beta)}} + C\frac{k^\alpha}{q^\alpha}\,.
\end{aligned}
$$

Using (29), (31) and Corollary 10 yields

$$
\begin{aligned}
\left\langle \lambda-\mu^{kq}, u^{hp}-\gamma(\lambda^{kq}+S u^{hp})\right\rangle_{\Gamma_C} &\le \left\|\lambda-\mu^{kq}\right\|_{L^2(\Gamma_C)}\left(\left\|u^{hp}\right\|_{L^2(\Gamma_C)} + \left\|\gamma\lambda^{kq}\right\|_{L^2(\Gamma_C)} + \left\|\gamma S u^{hp}\right\|_{L^2(\Gamma_C)}\right) \\
&\le \left\|\lambda-\mu^{kq}\right\|_{L^2(\Gamma_C)}\left(\left\|u^{hp}\right\|_{L^2(\Gamma_C)} + \gamma_0^{1/2}\frac{h^{(1+\beta)/2}}{p^{(2+\eta)/2}}\left\|\gamma^{1/2}\lambda^{kq}\right\|_{L^2(\Gamma_C)} + C\gamma_0\frac{h^\beta}{p^\eta}\left\|u^{hp}\right\|_{L^2(\Gamma)}\right) \\
&\le C\frac{k^\alpha}{q^\alpha}\|\lambda\|_{H^\alpha(\Gamma_C)}\left(\|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|\lambda\|_{\widetilde{H}^{-1/2}(\Gamma_C)} + C\gamma_0^{-1/2}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\|g\|_{H^{1+\alpha}(\Gamma_C)}\right) \\
&\le C\frac{k^\alpha}{q^\alpha} + \frac{C}{\gamma_0^{1/2}}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}}\frac{k^{1+2\alpha}}{q^{1+2\alpha}}\,.
\end{aligned}
$$

From the Cauchy-Schwarz inequality, (36) and Corollary 10 it follows that

$$
\begin{aligned}
-\left\langle \gamma(\mu^{kq}-\lambda^{kq}), E_{hp}u^{hp}\right\rangle_{\Gamma_C} &= \langle \gamma^{1/2}(\lambda^{kq}-\lambda), \gamma^{1/2}E_{hp}u^{hp}\rangle_{\Gamma_C} + \langle \gamma^{1/2}(\lambda-\mu^{kq}), \gamma^{1/2}E_{hp}u^{hp}\rangle_{\Gamma_C} \\
&\le C\gamma_0^{1/2}\frac{h^{\beta/2}}{p^{\eta/2}}\left(\left\|\gamma^{1/2}(\lambda^{kq}-\lambda)\right\|_{L^2(\Gamma_C)} + \gamma^{1/2}\left\|\lambda-\mu^{kq}\right\|_{L^2(\Gamma_C)}\right)\left\|u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \\
&\le C\gamma_0^{1/2}\frac{h^{\beta/2}}{p^{\eta/2}}\left(\left\|\gamma^{1/2}(\lambda^{kq}-\lambda)\right\|_{L^2(\Gamma_C)} + \gamma_0^{1/2}\frac{h^{(1+\beta)/2}}{p^{(2+\eta)/2}}\frac{k^\alpha}{q^\alpha}\|\lambda\|_{H^\alpha(\Gamma_C)}\right) \\
&\quad\cdot\left(\|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|\lambda\|_{\widetilde{H}^{-1/2}(\Gamma_C)} + C\gamma_0^{-1/2}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\|g\|_{H^{1+\alpha}(\Gamma_C)}\right) \\
&\le \left(\left\|\gamma^{1/2}(\lambda^{kq}-\lambda)\right\|_{L^2(\Gamma_C)} + \gamma_0^{1/2}\frac{h^{(1+\beta)/2}}{p^{(2+\eta)/2}}\frac{k^\alpha}{q^\alpha}C\right)\left(C\gamma_0^{1/2}\frac{h^{\beta/2}}{p^{\eta/2}} + C\frac{p}{h^{1/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\right) \\
&\le C\left(\gamma_0^{1/2}\frac{h^{\beta/2}}{p^{\eta/2}} + \frac{p}{h^{1/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\right)\left\|\gamma^{1/2}(\lambda^{kq}-\lambda)\right\|_{L^2(\Gamma_C)} + C\gamma_0\frac{h^{1/2+\beta}}{p^{1+\eta}}\frac{k^\alpha}{q^\alpha} + C\gamma_0^{1/2}\frac{h^{\beta/2}}{p^{\eta/2}}\frac{k^{1+2\alpha}}{q^{1+2\alpha}}\,.
\end{aligned}
$$

Furthermore, we obtain

$$
\begin{aligned}
-\left\langle \gamma(\lambda^{kq}+S_{hp}u^{hp}),\, E_{hp}(u^{hp}-v^{hp})\right\rangle_{\Gamma_C} &= \left\langle \gamma(-\lambda^{kq}+\lambda-\lambda-S_{hp}u^{hp}+S u^{hp}-S u^{hp}), E_{hp}(u^{hp}-v^{hp})\right\rangle_{\Gamma_C} \\
&= \left\langle \gamma(\lambda-\lambda^{kq}), E_{hp}(u^{hp}-v^{hp})\right\rangle_{\Gamma_C} + \left\langle \gamma(-\lambda-S u^{hp}), E_{hp}(u^{hp}-v^{hp})\right\rangle_{\Gamma_C} + \left\langle \gamma E_{hp}u^{hp}, E_{hp}(u^{hp}-v^{hp})\right\rangle_{\Gamma_C} \\
&=: A+B+D\,.
\end{aligned}
$$

From the Cauchy-Schwarz inequality, Young's inequality and (35) follows

$$
\begin{aligned}
A &\le \epsilon\left\|\gamma^{1/2}(\lambda-\lambda^{kq})\right\|^2_{L^2(\Gamma_C)} + \frac{1}{4\epsilon}\left\|\gamma^{1/2}E_{hp}(u^{hp}-v^{hp})\right\|^2_{L^2(\Gamma_C)} \\
&\le \epsilon\left\|\gamma^{1/2}(\lambda-\lambda^{kq})\right\|^2_{L^2(\Gamma_C)} + \frac{\gamma_0 C^2}{2\epsilon}\frac{h^\beta}{p^\eta}\left\|u^{hp}-u\right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \frac{\gamma_0 C^2}{2\epsilon}\frac{h^{1+2\alpha+\beta}}{p^{1+2\alpha+\eta}}\|u\|^2_{H^{1+\alpha}(\Gamma)} \\
&\le \epsilon\left\|\gamma^{1/2}(\lambda-\lambda^{kq})\right\|^2_{L^2(\Gamma_C)} + \frac{\gamma_0 C}{\epsilon}\frac{h^\beta}{p^\eta}\left\|u^{hp}-u\right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \frac{\gamma_0 C}{\epsilon}\frac{h^{1+2\alpha+\beta}}{p^{1+2\alpha+\eta}}\,.
\end{aligned}
$$

From $\lambda = -S u|_{\Gamma_C}$, the Cauchy-Schwarz inequality, (33), (34) and (35), Young's inequality and triangle inequality follows

$$
\begin{aligned}
B &= \left\langle \gamma^{1/2} S(u - v^{hp}) + \gamma^{1/2} S(v^{hp} - u^{hp}), \gamma^{1/2} E_{hp}(u^{hp} - v^{hp}) \right\rangle_{\Gamma_C} \\
&\leq \left(\left\| \gamma^{1/2} S(u - v^{hp}) \right\|_{L^2(\Gamma_C)} + \left\| \gamma^{1/2} S(v^{hp} - u^{hp}) \right\|_{L^2(\Gamma_C)} \right) \left\| \gamma^{1/2} E_{hp}(u^{hp} - v^{hp}) \right\|_{L^2(\Gamma_C)} \\
&\leq \left(\gamma_0^{1/2} C \frac{h^{(1+\beta)/2+\alpha}}{p^{(2+\eta)/2+\alpha}} \|u\|_{H^{1+\alpha}(\Gamma)} + C\gamma_0^{1/2} \frac{h^{\beta/2}}{p^{\eta/2}} \left\| v^{hp} - u^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \right) C\gamma_0^{1/2} \frac{h^{\beta/2}}{p^{\eta/2}} \left\| v^{hp} - u^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \\
&\leq \frac{\gamma_0 C^2}{4\epsilon} \frac{h^{1+\beta+2\alpha}}{p^{2+\eta+2\alpha}} \|u\|^2_{H^{1+\alpha}(\Gamma)} + (1+\epsilon) C^2 \gamma_0 \frac{h^\beta}{p^\eta} \left\| v^{hp} - u^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \\
&\leq \left(\frac{\gamma_0 C^2}{4\epsilon} \frac{h^{1+\beta+2\alpha}}{p^{2+\eta+2\alpha}} + 2(1+\epsilon) C^2 \gamma_0 \frac{h^{1+\beta+2\alpha}}{p^{1+\eta+2\alpha}} \right) \|u\|^2_{H^{1+\alpha}(\Gamma)} + 2(1+\epsilon) C^2 \gamma_0 \frac{h^\beta}{p^\eta} \left\| u - u^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \\
&\leq (1 + \epsilon + \epsilon^{-1}) C\gamma_0 \frac{h^{1+\beta+2\alpha}}{p^{1+\eta+2\alpha}} + (1+\epsilon) C\gamma_0 \frac{h^\beta}{p^\eta} \left\| u - u^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \; .
\end{aligned}
$$

From the Cauchy-Schwarz inequality, (36) and Young's inequality follows

$$
\begin{aligned}
D &\leq \gamma_0 \frac{h^{1+\beta}}{p^{2+\eta}} \left\| E_{hp} u^{hp} \right\|_{L^2(\Gamma_C)} \left\| E_{hp}(u^{hp} - v^{hp}) \right\|_{L^2(\Gamma_C)} \leq \gamma_0 C \frac{h^\beta}{p^\eta} \left\| u^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \left\| u^{hp} - v^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \\
&\leq \gamma_0 \frac{h^\beta}{p^\eta} C \left((1+\epsilon) \left\| u - u^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \frac{1}{4\epsilon} \left\| u - v^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \left\| u - v^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \left\| u - u^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \right) \\
&\leq \gamma_0 C \frac{h^\beta}{p^\eta} \left((1+\epsilon) \left\| u - u^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \left\| u - u^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \frac{1}{4\epsilon} \frac{h^{1+2\alpha}}{p^{1+2\alpha}} \|u\|^2_{H^{1+\alpha}(\Gamma)} + \frac{h^{1/2+\alpha}}{p^{1/2+\alpha}} \|u\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \|u\|_{H^{1+\alpha}(\Gamma)} \right) \\
&\leq (1+\epsilon) \gamma_0 C \frac{h^\beta}{p^\eta} \left\| u - u^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \gamma_0 C \frac{h^\beta}{p^\eta} \left\| u - u^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + (1+\epsilon^{-1}) \gamma_0 C \frac{h^{1/2+\beta+\alpha}}{p^{1/2+\eta+\alpha}} \; .
\end{aligned}
$$

Using $\lambda = -S u|_{\Gamma_C}$, the Cauchy-Schwarz inequality, Young's inequality and (33) yield

$$
\begin{aligned}
\left\langle \gamma(\lambda - \lambda^{kq}), \lambda + S v^{hp} \right\rangle_{\Gamma_C} &= \left\langle \gamma^{1/2}(\lambda - \lambda^{kq}), \gamma^{1/2} S(v^{hp} - u) \right\rangle_{\Gamma_C} \leq \epsilon \left\| \gamma^{1/2}(\lambda - \lambda^{kq}) \right\|^2_{L^2(\Gamma_C)} + \frac{1}{4\epsilon} \left\| \gamma^{1/2} S(v^{hp} - u) \right\|^2_{L^2(\Gamma_C)} \\
&\leq \epsilon \left\| \gamma^{1/2}(\lambda - \lambda^{kq}) \right\|^2_{L^2(\Gamma_C)} + \frac{C\gamma_0}{\epsilon} \frac{h^{1+\beta+2\alpha}}{p^{2+\eta+2\alpha}} \|u\|^2_{H^{1+\alpha}(\Gamma)} = \epsilon \left\| \gamma^{1/2}(\lambda - \lambda^{kq}) \right\|^2_{L^2(\Gamma_C)} + \frac{C\gamma_0}{\epsilon} \frac{h^{1+\beta+2\alpha}}{p^{2+\eta+2\alpha}} \; .
\end{aligned}
$$

Using $\lambda = -S u|_{\Gamma_C}$, the Cauchy-Schwarz inequality, Young's inequality, (33) and (34) yield

$$
\begin{aligned}
\left\langle \gamma(\lambda + S u^{hp}), S(u^{hp} - v^{hp}) \right\rangle_{\Gamma_C} &= \left\langle \gamma^{1/2} S(u^{hp} - v^{hp}) + \gamma^{1/2} S(v^{hp} - u), \gamma^{1/2} S(u^{hp} - v^{hp}) \right\rangle_{\Gamma_C} \\
&\leq \left\| \gamma^{1/2} S(u^{hp} - v^{hp}) \right\|^2_{L^2(\Gamma_C)} + \left\| \gamma^{1/2} S(v^{hp} - u) \right\|_{L^2(\Gamma_C)} \left\| \gamma^{1/2} S(u^{hp} - v^{hp}) \right\|_{L^2(\Gamma_C)} \\
&\leq (1+\epsilon) \left\| \gamma^{1/2} S(u^{hp} - v^{hp}) \right\|^2_{L^2(\Gamma_C)} + \frac{1}{4\epsilon} \left\| \gamma^{1/2} S(v^{hp} - u) \right\|^2_{L^2(\Gamma_C)} \\
&\leq 2(1+\epsilon) C^2 \gamma_0 \frac{h^\beta}{p^\eta} \left\| u^{hp} - u \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left(2(1+\epsilon) C^2 \gamma_0 p + \frac{C^2 \gamma_0}{\epsilon} \right) \frac{h^{1+\beta+2\alpha}}{p^{2+\eta+2\alpha}} \|u\|^2_{H^{1+\alpha}(\Gamma)} \\
&\leq (1+\epsilon) C\gamma_0 \frac{h^\beta}{p^\eta} \left\| u^{hp} - u \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + (1 + \epsilon + \epsilon^{-1}) C\gamma_0 \frac{h^{1+\beta+2\alpha}}{p^{1+\eta+2\alpha}} \; .
\end{aligned}
$$

Analogously to the estimate of part D we obtain

$$
\begin{aligned}
&- \left\langle \gamma E_{hp} u^{hp}, S(u^{hp} - v^{hp}) \right\rangle_{\Gamma_C} \leq \gamma_0 \frac{h^{1+\beta}}{p^{2+\eta}} \left\| E_{hp} u^{hp} \right\|_{L^2(\Gamma_C)} \left\| S(u^{hp} - v^{hp}) \right\|_{L^2(\Gamma_C)} \leq \gamma_0 C \frac{h^\beta}{p^\eta} \left\| u^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \left\| u^{hp} - v^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \\
&\leq (1+\epsilon) \gamma_0 C \frac{h^\beta}{p^\eta} \left\| u - u^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \gamma_0 C \frac{h^\beta}{p^\eta} \left\| u - u^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + (1+\epsilon^{-1}) \gamma_0 C \frac{h^{1/2+\beta+\alpha}}{p^{1/2+\eta+\alpha}} \; .
\end{aligned}
$$

Putting all these estimates together yields

$$
\begin{aligned}
&\left(\alpha_W - 2\epsilon_1 - (1+\epsilon+\epsilon^{-1})\gamma_0 C\frac{h^\beta}{p^\eta}\right)\|u-u^{hp}\|^2_{\widetilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)} + (\alpha_V - \epsilon_2)\|\psi-\psi^{hp}\|^2_{H^{-\frac{1}{2}}(\Gamma)} + (1-3\epsilon)\|\gamma^{\frac{1}{2}}(\lambda-\lambda^{kq})\|^2_{L^2(\Gamma_C)} \\
&\quad - \gamma_0 C\frac{h^\beta}{p^\eta}\left\|u-u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} - C\left(\gamma_0^{1/2}\frac{h^{\beta/2}}{p^{\eta/2}} + \frac{p}{h^{1/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\right)\left\|\gamma^{1/2}(\lambda^{kq}-\lambda)\right\|_{L^2(\Gamma_C)} \\
&\le \left(\epsilon_1^{-1}+\epsilon_2^{-1}+1\right)C\frac{h^{1+2\alpha}}{p^{1+2\alpha}} + (1+\epsilon+\epsilon^{-1})C\gamma_0\frac{h^{1+\beta+2\alpha}}{p^{1+\eta+2\alpha}} + (1+\epsilon^{-1})\gamma_0 C\frac{h^{1/2+\beta+\alpha}}{p^{1/2+\eta+\alpha}} + \frac{C\gamma_0}{\epsilon}\frac{h^{1+\beta+2\alpha}}{p^{2+\eta+2\alpha}} + \frac{C}{\epsilon\gamma_0}\frac{h^{1+2\alpha-\beta}}{p^{2\alpha-\eta}} \\
&\quad \frac{C}{\gamma_0^{1/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}} + \frac{C}{\gamma_0}\frac{k^{2+2\alpha}}{q^{2+2\alpha}}\frac{p^{2+\eta}}{h^{1+\beta}} + \frac{C}{\gamma_0^{1/2}}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}}\frac{k^{1+2\alpha}}{q^{1+2\alpha}} + C\gamma_0\frac{h^{1/2+\beta}}{p^{1+\eta}}\frac{k^\alpha}{q^\alpha} + C\gamma_0^{1/2}\frac{h^{\beta/2}}{p^{\eta/2}}\frac{k^{1+2\alpha}}{q^{1+2\alpha}} + C\frac{k^\alpha}{q^\alpha}\,.
\end{aligned}
$$

For ϵ, ϵ_1, ϵ_2 and γ_0 sufficiently small and $h \le 1$, $p \ge 1$, $\beta, \eta \ge 0$ and compressing the constants to the positive, generic ones yields that the error estimate has the following form

$$
\begin{aligned}
&C_1^2\|u-u^{hp}\|^2_{\widetilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)} - C_2\frac{h^\beta}{p^\eta}\left\|u-u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + C_4^2\|\gamma^{\frac{1}{2}}(\lambda-\lambda^{kq})\|^2_{L^2(\Gamma_C)} - C_5\left(\frac{h^{\beta/2}}{p^{\eta/2}} + \frac{p}{h^{1/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\right)\left\|\gamma^{1/2}(\lambda^{kq}-\lambda)\right\|_{L^2(\Gamma_C)} \\
&\quad + C_3^2\|\psi-\psi^{hp}\|^2_{H^{-\frac{1}{2}}(\Gamma)} \\
&= \left(C_1\|u-u^{hp}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)} - \frac{C_2}{2C_1}\frac{h^\beta}{p^\eta}\right)^2 + \left(C_4\|\gamma^{\frac{1}{2}}(\lambda-\lambda^{kq})\|_{L^2(\Gamma_C)} - \frac{C_5}{2C_4}\left(\frac{h^{\beta/2}}{p^{\eta/2}} + \frac{p}{h^{1/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\right)\right)^2 + C_3^2\|\psi-\psi^{hp}\|^2_{H^{-\frac{1}{2}}(\Gamma)} \\
&\quad - \frac{C_2^2}{4C_1^2}\frac{h^{2\beta}}{p^{2\eta}} - \frac{C_5^2}{4C_4^2}\left(\frac{h^{\beta/2}}{p^{\eta/2}} + \frac{p}{h^{1/2}}\frac{k^{1+\alpha}}{q^{1+\alpha}}\right)^2 \\
&\le C\left(\frac{h^{1/2+\beta+\alpha}}{p^{1/2+\eta+\alpha}} + \frac{h^{1+2\alpha-\beta}}{p^{2\alpha-\eta}} + \frac{k^{1+\alpha}}{q^{1+\alpha}}\frac{p^{(2+\eta)/2}}{h^{(1+\beta)/2}} + \frac{k^{2+2\alpha}}{q^{2+2\alpha}}\frac{p^{2+\eta}}{h^{1+\beta}} + \frac{k^\alpha}{q^\alpha}\right),
\end{aligned}
$$

where the equality sign results from the second binomial formula. From that we deduce the assertion by using the trivial lower bound of squared terms, taking the square root, separating the error terms and the convergence rates, squaring the resulting inequality and finally using once more Young's inequality.

Remark 17.

1. *For $h = k$, $p = q$, the a priori error estimate in Theorem 16 becomes*

$$
\|u-u^{hp}\|^2_{\widetilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)} + \|\gamma^{\frac{1}{2}}(\lambda-\lambda^{kq})\|^2_{L^2(\Gamma_C)} + \|\psi-\psi^{hp}\|^2_{H^{-\frac{1}{2}}(\Gamma)} \le C\left(\frac{h^\beta}{p^\eta} + \frac{h^\alpha}{p^\alpha} + \frac{h^{1/2+\alpha-\beta/2}}{p^{\alpha-\eta/2}}\right),
$$

   which is maximal for a range of β, e.g. $\beta = 1$ or $\beta = \alpha$, as $\alpha \in [0, 1/2)$, and $\eta = 2\alpha/3$, i.e. the error reduces at least like $h^{\alpha/2}p^{-\alpha/3}$.
2. *The term* (37) *and Lemma 14 require that the stabilization does not go too fast to zero. On the other hand, a closer inspection of the proof shows that the approximation of the Steklov-Poincare operator in the stabilization term requires $\beta, \eta > 0$ (i.e. γ scales "better" then hp^{-2}). Only in this case, the proof gives a convergence rate for $\left\|\gamma^{1/2}E_{hp}w^{hp}\right\|_{L^2(\Gamma_C)}$.*

For the conforming approximation of (13) by Bernstein polynomials $\lambda^{kq} \in M^+_{kq}(\mathcal{F}) \subset M^+(\mathcal{F})$ the properties of a corresponding interpolation operator into $M^+_{kq}(\mathcal{F})$ do not seem to be available in the literature. Assuming that a quasi-interpolation operator $\widetilde{\pi}_{M_{kq}} : L^2(\Gamma_C) \cap M^+(\mathcal{F}) \to M^+_{kq}(\mathcal{F})$ can be defined, such that $\widetilde{\pi}_{M_{kq}}$ satisfies the approximation property

$$
\left\|\eta - \widetilde{\pi}_{M_{kq}}\eta\right\|_{L^2(\Gamma_C)} \le C\left(\frac{k}{q}\right)^\alpha \|\eta\|_{H^\alpha(\Gamma_C)}\,, \tag{38}
$$

the proof of Theorem 16 yields:

Remark 18. *Let* $(u,\lambda) \in H^{1+\alpha}(\Gamma) \times H^{\alpha}(\Gamma_C) \cap C^0(\Gamma_C)$ *and* $(u^{hp}, \lambda^{kq}) \in \mathcal{V}_{hp} \times M^+_{kq}(\mathcal{F})$ *be the solutions of* (8), (13), *respectively, with* $g \geq 0$, $\mathcal{F}$ *linear, i.e.* $M^+_{kq}(\mathcal{F}) \subset M^+(\mathcal{F})$ *and* $\alpha \in [0, \frac{1}{2})$. *Under the assumption that* (38) *holds, there exists a constant* $C > 0$ *independent of* h, p, k, q, $\beta \geq 0$ *and* $\eta \geq 0$ *such that*

$$\|u - u^{hp}\|^2_{\widetilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)} + \|\gamma^{\frac{1}{2}}(\lambda - \lambda^{kq})\|^2_{L^2(\Gamma_C)} + \|\psi - \psi^{hp}\|^2_{H^{-\frac{1}{2}}(\Gamma)} \leq C\left(\frac{h^{1+2\alpha-\beta}}{p^{2\alpha-\eta}} + \frac{h^\beta}{p^\eta} + \frac{k^\alpha}{q^\alpha}\right),$$

with ψ, ψ^{hp} *given in* (15).

Proof. The proof is analogous to the proof of Theorem 16 except that we now use Corollary 11 instead of Corollary 10 and choose $\mu = \lambda^{kq} \in M^+_{kq}(\mathcal{F}) \subset M^+(\mathcal{F})$ to obtain

$$\inf_{\mu \in M^+(\mathcal{F})} \left\langle \lambda^{kq} - \mu, u \right\rangle_{\Gamma_C} + \left\langle g, \mu_n^{kq} - \lambda_n^{kq} + \mu_n - \lambda_n \right\rangle_{\Gamma_C} \leq \left\langle g, \mu_n^{kq} - \lambda_n \right\rangle_{\Gamma_C} \leq C \frac{k^\alpha}{q^\alpha} \|g\|_{L^2(\Gamma_C)} \|\lambda\|_{H^\alpha(\Gamma_C)} ,$$

as $\mu^{kq} = \widetilde{\pi}_{M_{kq}} \lambda$.

5. A posteriori error estimates

In this section we present an a posteriori error estimate of residual type for the mixed hp-BEM scheme which is independent of the selected discretization for λ_{kq}.

Lemma 19. *Let* (u,λ), (u^{hp}, λ^{kq}) *be the solution of* (8), (13) *respectively. Then there holds*

$$\begin{aligned}\left\langle \lambda - \lambda^{kq}, u^{hp} - u \right\rangle_{\Gamma_C} \leq & \left\langle \left(\lambda_n^{kq}\right)^+, \left(g - u_n^{hp}\right)^+ \right\rangle_{\Gamma_C} + \left\|\lambda^{kq} - \lambda\right\|_{\widetilde{H}^{-1/2}(\Gamma_C)} \left\|\left(g - u_n^{hp}\right)^-\right\|_{H^{1/2}(\Gamma_C)} + \left\|\left(\lambda_n^{kq}\right)^-\right\|_{\widetilde{H}^{-1/2}(\Gamma_C)} \left\|u^{hp} - u\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} \\ & + \left\|\left(\left|\lambda_t^{kq}\right| - \mathcal{F}\right)^+\right\|_{\widetilde{H}^{-1/2}(\Gamma_C)} \left\|u - u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} - \left\langle \left(\left|\lambda_t^{kq}\right| - \mathcal{F}\right)^-, \left|u_t^{hp}\right| \right\rangle_{\Gamma_C} \\ & - \left\langle \lambda_t^{kq}, u_t^{hp} \right\rangle_{\Gamma_C} + \left\langle \left|\lambda_t^{kq}\right|, \left|u_t^{hp}\right| \right\rangle_{\Gamma_C} ,\end{aligned}$$

where $v^+ = \max\{0, v\}$ *and* $v^- = \min\{0, v\}$, *i.e.* $v = v^+ + v^-$.

Proof. *Utilizing that* $\langle \lambda_n, u_n - g \rangle_{\Gamma_C} = 0$ *by* (8b), $u_n - g \leq 0$ *almost everywhere in* Γ_C *and* $\left(\lambda_n^{kq}\right)^+ \in L^2(\Gamma_C)$, *there holds*

$$\begin{aligned}\left\langle \lambda_n - \lambda_n^{kq}, u_n^{hp} - u_n \right\rangle_{\Gamma_C} &= \left\langle \lambda_n - \left(\lambda_n^{kq}\right)^+, u_n^{hp} - g \right\rangle_{\Gamma_C} + \langle \lambda_n, g - u_n \rangle_{\Gamma_C} - \left\langle \left(\lambda_n^{kq}\right)^+, g - u_n \right\rangle_{\Gamma_C} - \left\langle \left(\lambda_n^{kq}\right)^-, u_n^{hp} - u_n \right\rangle_{\Gamma_C} \\ &\leq \left\langle \lambda_n - \left(\lambda_n^{kq}\right)^+, u_n^{hp} - g \right\rangle_{\Gamma_C} - \left\langle \left(\lambda_n^{kq}\right)^-, u_n^{hp} - u_n \right\rangle_{\Gamma_C} ,\end{aligned}$$

and with $\lambda \in M^+(\mathcal{F})$

$$\begin{aligned}\left\langle \lambda_n - \left(\lambda_n^{kq}\right)^+, u_n^{hp} - g \right\rangle_{\Gamma_C} &= \left\langle \left(\lambda_n^{kq}\right)^+, g - u_n^{hp} \right\rangle_{\Gamma_C} + \left\langle -\lambda_n, \left(g - u_n^{hp}\right)^+ + \left(g - u_n^{hp}\right)^- \right\rangle_{\Gamma_C} \\ &\leq \left\langle \left(\lambda_n^{kq}\right)^+, g - u_n^{hp} \right\rangle_{\Gamma_C} + \left\langle \lambda_n^{kq} - \lambda_n - \left(\lambda_n^{kq}\right)^+ - \left(\lambda_n^{kq}\right)^-, \left(g - u_n^{hp}\right)^- \right\rangle_{\Gamma_C} \\ &= \left\langle \left(\lambda_n^{kq}\right)^+, \left(g - u_n^{hp}\right)^+ \right\rangle_{\Gamma_C} + \left\langle \lambda_n^{kq} - \lambda_n, \left(g - u_n^{hp}\right)^- \right\rangle_{\Gamma_C} - \left\langle \left(\lambda_n^{kq}\right)^-, \left(g - u_n^{hp}\right)^- \right\rangle_{\Gamma_C} \\ &\leq \left\langle \left(\lambda_n^{kq}\right)^+, \left(g - u_n^{hp}\right)^+ \right\rangle_{\Gamma_C} + \left\langle \lambda_n^{kq} - \lambda_n, \left(g - u_n^{hp}\right)^- \right\rangle_{\Gamma_C} .\end{aligned}$$

Application of Cauchy-Schwarz inequality and trivial estimates of the norms yields

$$\left\langle \lambda_n - \lambda_n^{kq}, u_n^{hp} - u_n \right\rangle_{\Gamma_C} \leq \left\langle \left(\lambda_n^{kq}\right)^+, \left(g - u_n^{hp}\right)^+ \right\rangle_{\Gamma_C} + \left\|\lambda^{kq} - \lambda\right\|_{\widetilde{H}^{-1/2}(\Gamma_C)} \left\|\left(g - u_n^{hp}\right)^-\right\|_{H^{1/2}(\Gamma_C)} + \left\|\left(\lambda_n^{kq}\right)^-\right\|_{\widetilde{H}^{-1/2}(\Gamma_C)} \left\|u^{hp} - u\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} .$$

For the tangential component we exploit $\langle\lambda_t, u_t\rangle_{\Gamma_C} = \langle\mathcal{F}, |u_t|\rangle_{\Gamma_C}$, $\left\langle\lambda_t, u_t^{hp}\right\rangle_{\Gamma_C} \le \left\langle\mathcal{F}, \left|u_t^{hp}\right|\right\rangle_{\Gamma_C}$, $v = v^+ + v^-$ *and the triangle inequality to obtain*

$$
\begin{aligned}
\left\langle\lambda_t - \lambda_t^{kq}\ , u_t^{hp} - u_t\right\rangle_{\Gamma_C} &\le \langle-\mathcal{F}, |u_t|\rangle_{\Gamma_C} + \left\langle\lambda_t^{kq}, u_t\right\rangle_{\Gamma_C} + \left\langle\mathcal{F}, \left|u_t^{hp}\right|\right\rangle_{\Gamma_C} - \left\langle\lambda_t^{kq}, u_t^{hp}\right\rangle_{\Gamma_C} \\
&\le \left\langle\left(\left|\lambda_t^{kq}\right| - \mathcal{F}\right)^+, |u_t|\right\rangle_{\Gamma_C} + \left\langle\mathcal{F}, \left|u_t^{hp}\right|\right\rangle_{\Gamma_C} - \left\langle\lambda_t^{kq}, u_t^{hp}\right\rangle_{\Gamma_C} \\
&\le \left\langle\left(\left|\lambda_t^{kq}\right| - \mathcal{F}\right)^+, \left|u_t - u_t^{hp}\right|\right\rangle_{\Gamma_C} + \left\langle\left(\left|\lambda_t^{kq}\right| - \mathcal{F}\right)^+ + \mathcal{F}, \left|u_t^{hp}\right|\right\rangle_{\Gamma_C} - \left\langle\lambda_t^{kq}, u_t^{hp}\right\rangle_{\Gamma_C} \\
&= \left\langle\left(\left|\lambda_t^{kq}\right| - \mathcal{F}\right)^+, \left|u_t - u_t^{hp}\right|\right\rangle_{\Gamma_C} - \left\langle\left(\left|\lambda_t^{kq}\right| - \mathcal{F}\right)^-, \left|u_t^{hp}\right|\right\rangle_{\Gamma_C} - \left\langle\lambda_t^{kq}, u_t^{hp}\right\rangle_{\Gamma_C} + \left\langle\left|\lambda_t^{kq}\right|, \left|u_t^{hp}\right|\right\rangle_{\Gamma_C} \\
&\le \left\|\left(\left|\lambda_t^{kq}\right| - \mathcal{F}\right)^+\right\|_{\widetilde{H}^{-1/2}(\Gamma_C)} \left\|u - u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} - \left\langle\left(\left|\lambda_t^{kq}\right| - \mathcal{F}\right)^-, \left|u_t^{hp}\right|\right\rangle_{\Gamma_C} - \left\langle\lambda_t^{kq}, u_t^{hp}\right\rangle_{\Gamma_C} + \left\langle\left|\lambda_t^{kq}\right|, \left|u_t^{hp}\right|\right\rangle_{\Gamma_C} .
\end{aligned}
$$

Lemma 20. *Let* (u, λ), (u^{hp}, λ^{kq}) *be the solution of* (8), (13) *respectively. Then there exists a constant* C *independent of* h, p, k *and* q *such that*

$$
\begin{aligned}
C\left(\left\|u - u^{hp}\right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left\|\psi - \psi^{hp}\right\|^2_{H^{-1/2}(\Gamma)}\right) &\le \sum_{E\in\mathcal{T}_h|_{\Gamma_N}} \frac{h_E}{p_E}\left\|f - S_{hp}u^{hp}\right\|^2_{L^2(E)} + \sum_{E\in\mathcal{T}_h|_{\Gamma_C}} \frac{h_E}{p_E}\left\|\lambda^{kq} + S_{hp}u^{hp}\right\|^2_{L^2(E)} \\
&+ \sum_{E\in\mathcal{T}_{h,\Gamma}} h_E \left\|\frac{\partial}{\partial s}\left(V\psi^{hp} - (K + \frac{1}{2})u^{hp}\right)\right\|^2_{L^2(E)} + \left\langle\left(\lambda_n^{kq}\right)^+, \left(g - u_n^{hp}\right)^+\right\rangle_{\Gamma_C} + \epsilon\left\|\lambda^{kq} - \lambda\right\|^2_{\widetilde{H}^{-1/2}(\Gamma_C)} + \frac{1}{4\epsilon}\left\|\left(g - u_n^{hp}\right)^-\right\|^2_{H^{1/2}(\Gamma_C)} \\
&+ \left\|\left(\lambda_n^{kq}\right)^-\right\|^2_{\widetilde{H}^{-1/2}(\Gamma_C)} + \left\|\left(\left|\lambda_t^{kq}\right| - \mathcal{F}\right)^+\right\|^2_{\widetilde{H}^{-1/2}(\Gamma_C)} - \left\langle\left(\left|\lambda_t^{kq}\right| - \mathcal{F}\right)^-, \left|u_t^{hp}\right|\right\rangle_{\Gamma_C} - \left\langle\lambda_t^{kq}, u_t^{hp}\right\rangle_{\Gamma_C} + \left\langle\left|\lambda_t^{kq}\right|, \left|u_t^{hp}\right|\right\rangle_{\Gamma_C} ,
\end{aligned}
$$

with $\epsilon > 0$ *arbitrary and* ψ, ψ^{hp} *given in* (15).

Proof. *Since* $u - u^{hp} \in \widetilde{H}^{1/2}(\Gamma_\Sigma)$ *there holds by Lemma 4*

$$
\begin{aligned}
C\left(\left\|u - u^{hp}\right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left\|\psi - \psi^{hp}\right\|^2_{H^{-1/2}(\Gamma)}\right) &\le \left\langle W(u - u^{hp}), u - u^{hp}\right\rangle_{\Gamma_\Sigma} + \left\langle V(\psi - \psi^{hp}), \psi - \psi^{hp}\right\rangle_\Gamma \\
&= \left\langle Su - S_{hp}u^{hp}, u - u^{hp}\right\rangle_{\Gamma_\Sigma} + \left\langle V(\psi^*_{hp} - \psi^{hp}), \psi - \psi^{hp}\right\rangle_\Gamma .
\end{aligned}
$$

From Lemma 8 and (8a) *it follows that*

$$
\begin{aligned}
\left\langle Su - S_{hp}u^{hp}, u - u^{hp}\right\rangle_{\Gamma_\Sigma} &= \left\langle Su - S_{hp}u^{hp}, u - u^{hp}\right\rangle_{\Gamma_\Sigma} + \langle Su - S_{hp}u^{hp}, u^{hp} - v^{hp}\rangle_{\Gamma_\Sigma} + \left\langle\lambda - \lambda^{kq}, u^{hp} - v^{hp}\right\rangle_{\Gamma_C} \\
&\quad + \left\langle\gamma(\lambda^{kq} + S_{hp}u^{hp}), S_{hp}(u^{hp} - v^{hp})\right\rangle_{\Gamma_C} \\
&= \left\langle f - S_{hp}u^{hp}, u - v^{hp}\right\rangle_{\Gamma_N} + \left\langle-\lambda^{kq} - S_{hp}u^{hp}, u - v^{hp}\right\rangle_{\Gamma_C} + \left\langle\lambda - \lambda^{kq}, u^{hp} - u\right\rangle_{\Gamma_C} + \left\langle\gamma(\lambda^{kq} + S_{hp}u^{hp}), S_{hp}(u^{hp} - v^{hp})\right\rangle_{\Gamma_C} .
\end{aligned}
$$

Let I_{hp} *be the Clement-interpolation operator mapping onto* $\mathcal{V}_{hp}$ *with the property (see [28] and interpolation between* L^2 *and* H^1*)*

$$
\left\|v - I_{hp}v\right\|_{L^2(E)} \le C\left(\frac{h_E}{p_E}\right)^{1/2} \|v\|_{H^{1/2}(\omega(E))} ,
$$

with $\omega(E)$ *a net around* E*. Then, an application of the Cauchy-Schwarz inequality yields with* $v^{hp} := u^{hp} + I_{hp}(u - u^{hp})$

$$
\begin{aligned}
\left\langle f - S_{hp}u^{hp}, u - v^{hp}\right\rangle_{\Gamma_N} &\le C\sum_{E\in\mathcal{T}_h|_{\Gamma_N}}\left(\frac{h_E}{p_E}\right)^{1/2}\left\|f - S_{hp}u^{hp}\right\|_{L^2(E)}\left\|u - u^{hp}\right\|_{H^{1/2}(\omega(E))} , \\
\left\langle-\lambda^{kq} - S_{hp}u^{hp}, u - v^{hp}\right\rangle_{\Gamma_C} &\le C\sum_{E\in\mathcal{T}_h|_{\Gamma_C}}\left(\frac{h_E}{p_E}\right)^{1/2}\left\|\lambda^{kq} + S_{hp}u^{hp}\right\|_{L^2(E)}\left\|u - u^{hp}\right\|_{H^{1/2}(\omega(E))} .
\end{aligned}
$$

Since $u_{hp} \in \mathcal{V}_{hp} \subset H^1_0(\Gamma_\Sigma)$ and $\psi_{hp} \in \mathcal{V}^D_{hp} \subset L^2(\Gamma)$, the mapping properties of V and K [11] yield

$$V(\psi^{hp} - \psi^*_{hp}) = V\psi^{hp} - (K + \frac{1}{2})u^{hp} \in H^1(\Gamma) \subset C^0(\Gamma) \ .$$

Furthermore, $V(\psi^{hp} - \psi^*_{hp})$ is orthogonal in $L^2(\Gamma)$ to $\mathcal{V}^D_{hp}$, Lemma 4. Hence, for the characteristic function $\chi_E \in \mathcal{V}^D_{hp}$ of an element $E \in \mathcal{T}_{h,\Gamma}$ there holds

$$0 = \left\langle V(\psi^{hp} - \psi^*_{hp}), \chi_E \right\rangle_\Gamma = \int_E V(\psi^{hp} - \psi^*_{hp}) \, ds \ ,$$

and therefore the continuous function $V(\psi_{hp} - \psi^*_{hp})$ has a root on each boundary segment E. Since $V(\psi_{hp} - \psi^*_{hp}) \in H^1(\Gamma)$, the application of [7, Theorem 5.1] yields

$$\left\langle V(\psi^{hp} - \psi^*_{hp}), \psi - \psi^{hp} \right\rangle_\Gamma \leq \left\| V(\psi^{hp} - \psi^*_{hp}) \right\|_{H^{\frac{1}{2}}(\Gamma)} \left\| \psi^{hp} - \psi \right\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \left(\sum_{E \in \mathcal{T}_{h,\Gamma}} h_E \left\| \frac{\partial}{\partial s} (V(\psi^{hp} - \psi^*_{hp}) \right\|^2_{L^2(E)} \right)^{\frac{1}{2}} \left\| \psi^{hp} - \psi \right\|_{H^{-\frac{1}{2}}(\Gamma)} \ .$$

Since $v^{hp} = u^{hp} + I_{hp}(u - u^{hp})$, there holds by Cauchy-Schwarz inequality (twice), Theorem 5 and the $H^{1/2}$-stability of I_{hp} that

$$\begin{aligned}
\left\langle \gamma(\lambda^{kq} + S_{hp}u^{hp}), S_{hp}(u^{hp} - v^{hp}) \right\rangle_{\Gamma_C} &= \gamma_0 \sum_{E \in \mathcal{T}_h|_{\Gamma_C}} \int_E \left(\frac{h_E^{1/2+\beta}}{p_E^{1+\eta}} \right) (\lambda^{kq} + S_{hp}u^{hp}) \left(\frac{h_E^{1/2}}{p_E} \right) S_{hp}(I_{hp}(u^{hp} - u)) \, ds \\
&\leq \gamma_0 \left(\sum_{E \in \mathcal{T}_h|_{\Gamma_C}} \frac{h_E^{1+2\beta}}{p_E^{2+2\eta}} \left\| \lambda^{kq} + S_{hp}u^{hp} \right\|^2_{L^2(E)} \right)^{\frac{1}{2}} \left(\sum_{E \in \mathcal{T}_h|_{\Gamma_C}} \left\| \frac{h_E^{1/2}}{p_E} S_{hp}(I_{hp}(u^{hp} - u)) \right\|^2_{L^2(E)} \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{E \in \mathcal{T}_h|_{\Gamma_C}} \frac{h_E^{1+2\beta}}{p_E^{2+2\eta}} \left\| \lambda^{kq} + S_{hp}u^{hp} \right\|^2_{L^2(E)} \right)^{\frac{1}{2}} \left\| u - u^{hp} \right\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)} \ .
\end{aligned}$$

In total this yields with Lemma 19 that

$$\begin{aligned}
&C \left(\left\| u - u^{hp} \right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left\| \psi - \psi^{hp} \right\|^2_{H^{-1/2}(\Gamma)} \right) \leq \sum_{E \in \mathcal{T}_h|_{\Gamma_N}} \left(\frac{h_E}{p_E} \right)^{1/2} \left\| f - S_{hp}u^{hp} \right\|_{L^2(E)} \left\| u - u^{hp} \right\|_{H^{1/2}(\omega(E))} \\
&+ \sum_{E \in \mathcal{T}_h|_{\Gamma_C}} \left(\frac{h_E}{p_E} \right)^{1/2} \left\| \lambda^{kq} + S_{hp}u^{hp} \right\|_{L^2(E)} \left\| u - u^{hp} \right\|_{H^{1/2}(\omega(E))} + \left(\sum_{E \in \mathcal{T}_h|_{\Gamma_C}} \frac{h_E^{1+2\beta}}{p_E^{2+2\eta}} \left\| \lambda^{kq} + S_{hp}u^{hp} \right\|^2_{L^2(E)} \right)^{\frac{1}{2}} \left\| u - u^{hp} \right\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)} \\
&+ \left(\sum_{E \in \mathcal{T}_{h,\Gamma}} h_E \left\| \frac{\partial}{\partial s} (V(\psi^{hp} - \psi^*_{hp}) \right\|^2_{L^2(E)} \right)^{\frac{1}{2}} \left\| \psi^{hp} - \psi \right\|_{H^{-\frac{1}{2}}(\Gamma)} + \left\langle \left(\lambda_n^{kq}\right)^+, \left(g - u_n^{hp}\right)^+ \right\rangle_{\Gamma_C} + \left\| \lambda^{kq} - \lambda \right\|_{\widetilde{H}^{-1/2}(\Gamma_C)} \left\| \left(g - u_n^{hp}\right)^- \right\|_{H^{1/2}(\Gamma_C)} \\
&+ \left\| \left(\lambda_n^{kq}\right)^- \right\|_{\widetilde{H}^{-1/2}(\Gamma_C)} \left\| u^{hp} - u \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left\| \left(\left|\lambda_t^{kq}\right| - \mathcal{F} \right)^+ \right\|_{\widetilde{H}^{-1/2}(\Gamma_C)} \left\| u - u^{hp} \right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} - \left\langle \left(\left|\lambda_t^{kq}\right| - \mathcal{F} \right)^-, \left|u_t^{hp}\right| \right\rangle_{\Gamma_C} \\
&- \left\langle \lambda_t^{kq}, u_t^{hp} \right\rangle_{\Gamma_C} + \left\langle \left|\lambda_t^{kq}\right|, \left|u_t^{hp}\right| \right\rangle_{\Gamma_C} \ .
\end{aligned}$$

The assertion follows with Young's inequality and $h \leq 1$, $p_E^{2+2\eta} \geq p_E$.

Lemma 21. *Let* (u,λ), (u^{hp},λ^{kq}) *be the solution of* (8), (13) *respectively. Then there holds*

$$\frac{\widetilde{\beta}}{C}\left\|\lambda-\lambda^{kq}\right\|_{\widetilde{H}^{-1/2}(\Gamma_C)} \leq \left\|u-u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left\|\psi-\psi^{hp}\right\|_{H^{-1/2}(\Gamma)} + \left(\sum_{E\in\mathcal{T}_h|_{\Gamma_C}} \frac{h_E}{p_E}\left\|\lambda^{kq}+S_{hp}u^{hp}\right\|^2_{L^2(E)}\right)^{\frac{1}{2}}$$
$$+\left(\sum_{E\in\mathcal{T}_h|_{\Gamma_N}} \frac{h_E}{p_E}\left\|f-S_{hp}u^{hp}\right\|^2_{L^2(E)}\right)^{1/2},$$

with ψ, ψ^{hp} *given in* (15).

Proof. *Let* $v\in\widetilde{H}^{1/2}(\Gamma_\Sigma)$ *and* $v^{hp}:=I_{hp}v\in\mathcal{V}_{hp}$, *then by Lemma 8 and* (8a) *there holds*

$$\begin{aligned}
\left\langle\lambda-\lambda^{kq},v\right\rangle_{\Gamma_C} &= \left\langle\lambda-\lambda^{kq},v-v^{hp}\right\rangle_{\Gamma_C} - \langle Su-S_{hp}u^{hp},v^{hp}\rangle_{\Gamma_\Sigma} - \left\langle\gamma(\lambda^{kq}+S_{hp}u^{hp}),S_{hp}v^{hp}\right\rangle_{\Gamma_C}\\
&= \left\langle f,v-v^{hp}\right\rangle_{\Gamma_N} - \left\langle Su,v-v^{hp}\right\rangle_{\Gamma_\Sigma} - \left\langle\lambda^{kq},v-v^{hp}\right\rangle_{\Gamma_C} - \langle Su-S_{hp}u^{hp},v^{hp}\rangle_{\Gamma_\Sigma} - \left\langle\gamma(\lambda^{kq}+S_{hp}u^{hp}),S_{hp}v^{hp}\right\rangle_{\Gamma_C}\\
&= \left\langle f-S_{hp}u^{hp},v-v^{hp}\right\rangle_{\Gamma_N} + \left\langle-\lambda^{kq}-S_{hp}u^{hp},v-v^{hp}\right\rangle_{\Gamma_C} - \left\langle Su-S_{hp}u^{hp},v\right\rangle_{\Gamma_\Sigma} - \left\langle\gamma(\lambda^{kq}+S_{hp}u^{hp}),S_{hp}v^{hp}\right\rangle_{\Gamma_C}.
\end{aligned}$$

For the third term we obtain by the definition of ψ *and* ψ^{hp} *in* (15) *and by the continuity of the operators that*

$$\begin{aligned}
\left\langle Su-S_{hp}u^{hp},v\right\rangle_{\Gamma_\Sigma} &= \left\langle W(u-u^{hp})+(K'+\frac{1}{2})(\psi-\psi^{hp}),v\right\rangle_{\Gamma_\Sigma}\\
&\leq C_W\left\|u-u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}\|v\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + (C_{K'}+\frac{1}{2})\left\|\psi-\psi^{hp}\right\|_{H^{-1/2}(\Gamma)}\|v\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}.
\end{aligned}$$

The first two and the last term can be handled as in Lemma 20, leading to

$$\frac{1}{C}\left\langle\lambda-\lambda^{kq},v\right\rangle_{\Gamma_C} \leq \left\|u-u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}\|v\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left\|\psi-\psi^{hp}\right\|_{H^{-1/2}(\Gamma)}\|v\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left(\sum_{E\in\mathcal{T}_h|_{\Gamma_C}}\frac{h_E^{1+2\beta}}{p_E^{2+2\eta}}\left\|\lambda^{kq}+S_{hp}u^{hp}\right\|^2_{L^2(E)}\right)^{\frac{1}{2}}\|v\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_\Sigma)}$$
$$+\sum_{E\in\mathcal{T}_h|_{\Gamma_N}}\left(\frac{h_E}{p_E}\right)^{1/2}\left\|f-S_{hp}u^{hp}\right\|_{L^2(E)}\|v\|_{H^{1/2}(\omega(E))} + \sum_{E\in\mathcal{T}_h|_{\Gamma_C}}\left(\frac{h_E}{p_E}\right)^{1/2}\left\|\lambda^{kq}+S_{hp}u^{hp}\right\|_{L^2(E)}\|v\|_{H^{1/2}(\omega(E))}.$$

The assertion follows from the continuous inf-sup condition (9) *and Cauchy-Schwarz inequality.*

Combining the two Lemmas 20 and 21 immediately yields the following theorem if $\epsilon>0$ in Lemma 20 is chosen sufficiently small.

Theorem 22 (Residual based error estimate). *Let* (u,λ), (u^{hp},λ^{kq}) *be the solution of* (8), (13) *respectively. Then there holds*

$$C\left(\left\|u-u^{hp}\right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left\|\psi-\psi^{hp}\right\|^2_{H^{-1/2}(\Gamma)} + \left\|\lambda^{kq}-\lambda\right\|^2_{\widetilde{H}^{-1/2}(\Gamma_C)}\right) \leq \sum_{E\in\mathcal{T}_h|_{\Gamma_N}}\frac{h_E}{p_E}\left\|f-S_{hp}u^{hp}\right\|^2_{L^2(E)} + \sum_{E\in\mathcal{T}_h|_{\Gamma_C}}\frac{h_E}{p_E}\left\|\lambda^{kq}+S_{hp}u^{hp}\right\|^2_{L^2(E)}$$
$$+\sum_{E\in\mathcal{T}_{h,\Gamma}} h_E\left\|\frac{\partial}{\partial s}\left(V\psi^{hp}-(K+\frac{1}{2})u^{hp}\right)\right\|^2_{L^2(E)} + \left\langle\left(\lambda_n^{kq}\right)^+,\left(g-u_n^{hp}\right)^+\right\rangle_{\Gamma_C} + \left\|\left(g-u_n^{hp}\right)^-\right\|^2_{H^{1/2}(\Gamma_C)}$$
$$+\left\|\left(\lambda_n^{kq}\right)^-\right\|^2_{\widetilde{H}^{-1/2}(\Gamma_C)} + \left\|\left(\left|\lambda_t^{kq}\right|-\mathcal{F}\right)^+\right\|^2_{\widetilde{H}^{-1/2}(\Gamma_C)} - \left\langle\left(\left|\lambda_t^{kq}\right|-\mathcal{F}\right)^-,\left|u_t^{hp}\right|\right\rangle_{\Gamma_C} - \left\langle\lambda_t^{kq},u_t^{hp}\right\rangle_{\Gamma_C} + \left\langle\left|\lambda_t^{kq}\right|,\left|u_t^{hp}\right|\right\rangle_{\Gamma_C},$$

with ψ, ψ^{hp} *given in* (15).

It is worth pointing out, that the stabilization implies no additional term in the a posteriori error estimate compared to the non-stabilized case in [3, Theorem 11] and does not even effect the scaling.

Corollary 23. *For $\lambda^{kq} \in M^{+}_{kq}(\mathcal{F})$ the estimate of Theorem 22 is reduced by non-conformity terms and simplifies the complementarity and stick error contributions.*

$$
\begin{aligned}
C\left(\left\|u-u^{hp}\right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}+\left\|\psi-\psi^{hp}\right\|^2_{H^{-1/2}(\Gamma)}+\left\|\lambda^{kq}-\lambda\right\|^2_{\widetilde{H}^{-1/2}(\Gamma_C)}\right) &\leq \sum_{E\in\mathcal{T}_h|_{\Gamma_N}}\frac{h_E}{p_E}\left\|f-S_{hp}u^{hp}\right\|^2_{L^2(E)}\\
+\sum_{E\in\mathcal{T}_h|_{\Gamma_C}}\frac{h_E}{p_E}\left\|\lambda^{kq}+S_{hp}u^{hp}\right\|^2_{L^2(E)}+\sum_{E\in\mathcal{T}_{h,\Gamma}}h_E\left\|\frac{\partial}{\partial s}\left(V\psi^{hp}-(K+\frac{1}{2})u^{hp}\right)\right\|^2_{L^2(E)}\\
+\left\langle\lambda^{kq}_n,\left(g-u^{hp}_n\right)^+\right\rangle_{\Gamma_C}+\left\|\left(g-u^{hp}_n\right)^-\right\|^2_{H^{1/2}(\Gamma_C)}-\left\langle\left|\lambda^{kq}_t\right|-\mathcal{F},\left|u^{hp}_t\right|\right\rangle_{\Gamma_C}-\left\langle\lambda^{kq}_t,u^{hp}_t\right\rangle_{\Gamma_C}+\left\langle\left|\lambda^{kq}_t\right|,\left|u^{hp}_t\right|\right\rangle_{\Gamma_C},
\end{aligned}
$$

with ψ, ψ^{hp} given in (15).

6. Implementational challenges

For the contact stabilized BEM (13) two non-standard matrices must be implemented, namely $\left\langle\gamma\lambda^{kq}, S_{hp}v^{hp}\right\rangle_{\Gamma_C}$ and $\left\langle\gamma S_{hp}u^{hp}, S_{hp}v^{hp}\right\rangle_{\Gamma_C}$. To restrain from additional difficulties we use the same mesh for λ^{kq} and u^{hp} on Γ_C. Hence, the singularities of $S_{hp}v^{hp}$ for the outer quadrature coincide with the nodes of the mesh for λ^{kq} and the standard outer quadrature technique for the BE-potentials can be applied. In the implementation we utilize the representations $S_{hp}v^{hp} = Wv^{hp} + (K+\frac{1}{2})^\top i_{hp}\chi^{hp}$ where $\chi^{hp} = V^{-1}_{hp}i^*_{hp}(K+\frac{1}{2})v^{hp} \in \mathcal{V}^D_{hp}$ and $Wv = -\frac{d}{ds}V^*\frac{dv}{ds}$, where V^* is the single layer potential with modified constants in the kernel function [33, p. 163]. Hence, performing integration by parts element-wise yields

$$
\begin{aligned}
\left\langle\gamma\lambda^{kq}, S_{hp}v^{hp}\right\rangle_{\Gamma_C} &= \left\langle\gamma\lambda^{kq}, -\frac{d}{ds}V^*\frac{d}{ds}v^{hp}\right\rangle_{\Gamma_C}+\left\langle\gamma\lambda^{kq},(K+\frac{1}{2})^\top\chi^{hp}\right\rangle_{\Gamma_C}\\
&=\gamma_0\sum_{E\in\mathcal{T}_h|_{\Gamma_C}}\frac{h_E}{p_E^2}\left\langle\frac{d}{ds}\lambda^{kq}, V^*\frac{d}{ds}v^{hp}\right\rangle_E-\left[\lambda^{kq}V^*\frac{d}{ds}v^{hp}\right]_{\partial E}+\left\langle\gamma\lambda^{kq},(K+\frac{1}{2})^\top\chi^{hp}\right\rangle_{\Gamma_C},
\end{aligned}
$$

where ∂E are the two endpoints of the interval E. Each of these terms can now be computed by standard BEM techniques, e.g. decomposition into far-field, near-field and self-element with the corresponding (hp-composite) Gauss-Quadrature for the outer integral and the analytic evaluation of the inner integral [26]. The algebraic representation of χ^{hp} for the computation of the standard Steklov-operator $\left\langle S_{hp}u^{hp}, v^{hp}\right\rangle_{\Gamma_\Sigma}$ is reused, i.e.

$$
\vec{\lambda}^\top\widetilde{\mathbf{S}}\vec{v} = \vec{\lambda}^\top\widetilde{\mathbf{W}}\vec{v}+\vec{\lambda}^\top\left(\widetilde{\mathbf{K}}+\widetilde{\mathbf{I}}/2\right)^\top\mathbf{V}^{-1}\left(\mathbf{K}+\mathbf{I}/2\right)\vec{v}\,.
$$

For the second matrix we obtain

$$
\begin{aligned}
\left\langle\gamma S_{hp}u^{hp}, S_{hp}v^{hp}\right\rangle_{\Gamma_C} &= \left\langle\gamma Wu^{hp}+\gamma(K+\frac{1}{2})^\top\zeta^{hp}, Wv^{hp}+(K+\frac{1}{2})^\top\chi^{hp}\right\rangle_{\Gamma_C}\\
=\left\langle\gamma Wu^{hp}, Wv^{hp}\right\rangle_{\Gamma_C}+\left\langle\gamma Wu^{hp},(K+\frac{1}{2})^\top\chi^{hp}\right\rangle_{\Gamma_C}&+\left\langle\gamma(K+\frac{1}{2})^\top\zeta^{hp}, Wv^{hp}\right\rangle_{\Gamma_C}+\left\langle\gamma(K+\frac{1}{2})^\top\zeta^{hp},(K+\frac{1}{2})^\top\chi^{hp}\right\rangle_{\Gamma_C}.
\end{aligned}
$$

Here, an element-wise integration by parts in the hypersingular integral operator yields no advantages, except for $\left\langle\gamma Wu^{hp},\chi^{hp}\right\rangle_{\Gamma_C}$ and $\left\langle\gamma\zeta^{hp}, Wv^{hp}\right\rangle_{\Gamma_C}$, and, therefore, the tangential derivative is approximated by a central finite difference quotient with a step length of 10^{-4} on the reference interval. This yields the matrix representation

$$
\begin{aligned}
\vec{v}^\top\widehat{\mathbf{S}}\vec{u} = \vec{v}^\top\widehat{\mathbf{W}\mathbf{W}}\vec{u}+\vec{v}^\top\left(\mathbf{K}+\frac{1}{2}\mathbf{I}\right)^\top\mathbf{V}^{-\top}\left(\widehat{\mathbf{W}\mathbf{K}^\top}+\frac{1}{2}\widehat{\mathbf{W}\mathbf{I}}\right)\vec{u}+\vec{v}^\top\left(\widehat{\mathbf{W}\mathbf{K}^\top}+\frac{1}{2}\widehat{\mathbf{W}\mathbf{I}}\right)^\top\mathbf{V}^{-1}\left(\mathbf{K}+\frac{1}{2}\mathbf{I}\right)\vec{u}\\
+\vec{v}^\top\left(\mathbf{K}+\frac{1}{2}\mathbf{I}\right)^\top\mathbf{V}^{-\top}\left(\widehat{\mathbf{K}^\top\mathbf{K}^\top}+\frac{1}{2}\widehat{\mathbf{K}^\top\mathbf{I}}+\frac{1}{2}\widehat{\mathbf{K}^\top\mathbf{I}}^\top+\frac{1}{4}\widehat{\mathbf{I}\mathbf{I}}\right)\mathbf{V}^{-1}\left(\mathbf{K}+\frac{1}{2}\mathbf{I}\right)\vec{u}\,.
\end{aligned}
$$

Most of the computational time is required for the matrices $\widehat{\mathbf{WW}}$, $\widehat{\mathbf{K}^\top\mathbf{K}^\top}$ and $\widehat{\mathbf{WK}^\top}$. Hence, their symmetry and other optimization possibilities should be exploited thoroughly. As an alternative to the approximation of the tangential derivatives by finite difference quotients one can approximate the function $V\frac{dv}{ds}$ by a polynomial and work with the derivative of this approximation.

7. Modifications for Coulomb friction

Tresca friction may yield unphysical behavior, namely non-zero tangential traction and stick-slip transition outside the actual contact zone. Therefore, in many applications the more realistic Coulomb friction is applied, in which the friction threshold $\mathcal{F}$ is replaced by $\mathcal{F}|\sigma_n(u)|$. In the discretization which we present here only the Lagrange multiplier set must be adapted, namely

$$M^+(\mathcal{F}\lambda_n) := \left\{\mu \in \widetilde{H}^{-1/2}(\Gamma_C) : \langle \mu, v\rangle_{\Gamma_C} \leq \langle \mathcal{F}\lambda_n, |v_t|\rangle_{\Gamma_C}\ \forall v \in \widetilde{H}^{1/2}(\Gamma_\Sigma), v_n \leq 0\right\}\ , \tag{39}$$

$$M^+_{kq}(\mathcal{F}\lambda_n^{kq}) := \left\{\mu^{kq} \in L^2(\Gamma_C) : \mu^{kq}|_E(x) = \sum_{i=0}^{q_E} \mu_i^E B_{i,q_E}(\Psi_E^{-1}(x))\ \forall E \in \widehat{\mathcal{T}}_k,\ (\mu_i^E)_n \geq 0,\ \left|(\mu_i^E)_t\right| \leq \mathcal{F}(\Psi_E(iq_E^{-1}))(\lambda_i^E)_n\right\}\ , \tag{40}$$

$$\widetilde{M}^+_{kq}(\mathcal{F}\lambda_n^{kq}) := \left\{\mu^{kq} \in L^2(\Gamma_C) : \mu^{kq}|_E \circ \Psi_E \in \left[\mathbb{P}_{q_E}([-1,1])\right]^2,\ \mu_n^{kq} \geq 0,\ -\mathcal{F}\lambda_n^{kq} \leq \mu_t^{kq} \leq \mathcal{F}\lambda_n^{kq} \text{ in } G_{kq}\right\}\ . \tag{41}$$

A standard iterative solver technique for Coulomb friction is to solve a sequence of Tresca frictional problems in which the friction threshold $\mathcal{F}\lambda_n$ of the current Tresca subproblem is obtained from the previous iterative Tresca solution. Since that solution is updated in the next Tresca iteration anyway we solve the subproblem inexactly by a single semi-smooth Newton step.

Theorem 24. *Let (u,λ), (u^{hp},λ^{kq}) be the solution of (8), (13) respectively, with the Lagrange multiplier sets modified according to Coulomb friction. Under the assumption that $\lambda_t = \mathcal{F}\lambda_n\xi$, $\xi \in Dir_t(u_t)$ where $Dir_t(u_t)$ is the subdifferential of the convex map $u_t \mapsto |u_t|$ (see [18]), $\mathcal{F} \geq 0$ constant and $\mathcal{F}\,\|\xi\|$ sufficiently small, there holds*

$$\begin{aligned}
&C\left(\left\|u-u^{hp}\right\|^2_{\widetilde{H}^{1/2}(\Gamma_\Sigma)} + \left\|\psi-\psi^{hp}\right\|^2_{H^{-1/2}(\Gamma)} + \left\|\lambda^{kq}-\lambda\right\|^2_{\widetilde{H}^{-1/2}(\Gamma_C)}\right) \leq \sum_{E\in\mathcal{T}_h|_{\Gamma_N}} \frac{h_E}{p_E}\left\|f - S_{hp}u^{hp}\right\|^2_{L^2(E)} + \sum_{E\in\mathcal{T}_h|_{\Gamma_C}} \frac{h_E}{p_E}\left\|\lambda^{kq} + S_{hp}u^{hp}\right\|^2_{L^2(E)}\\
&\quad + \sum_{E\in\mathcal{T}_{h,\Gamma}} h_E \left\|\frac{\partial}{\partial s}\left(V\psi^{hp} - (K+\frac{1}{2})u^{hp}\right)\right\|^2_{L^2(E)} + \left\|(\lambda^{kq})_n^-\right\|^2_{\widetilde{H}^{-1/2}(\Gamma_C)} + \left\|\left(|(\lambda^{kq})_t| - \mathcal{F}(\lambda^{kq})_n^+\right)^+\right\|_{\widetilde{H}^{-1/2}(\Gamma_C)}\\
&\quad - \left\langle\left(|(\lambda^{kq})_t| - \mathcal{F}(\lambda^{kq})_n^+\right)^-, \left|(u^{hp})_t\right|\right\rangle_{\Gamma_C} + \left\langle\left|(\lambda^{kq})_t\right|, \left|(u^{hp})_t\right|\right\rangle_{\Gamma_C} - \left\langle(\lambda^{kq})_t, (u^{hp})_t\right\rangle_{\Gamma_C}\\
&\quad + \left\|(g-(u^{hp})_n)^-\right\|_{H^{1/2}(\Gamma_C)} + \left\langle\left(\lambda^{kq}\right)_n^+, \left(g - \left(u^{hp}\right)_n\right)^+\right\rangle_{\Gamma_C}\Bigg)\ .
\end{aligned}$$

Proof. The same arguments as for Theorem 22 apply, only the estimate of the tangential component in Lemma 19 changes, [18, 3]. From $\lambda_t = \mathcal{F}\lambda_n\xi$ follows

$$\begin{aligned}
\left\langle\lambda_t - \lambda_t^{kq}, u_t^{hp} - u_t\right\rangle_{\Gamma_C} &= \left\langle\lambda_t^{kq} - \mathcal{F}\xi\left(\lambda^{kq}\right)_n, u_t - \left(u^{hp}\right)_t\right\rangle_{\Gamma_C} + \left\langle\mathcal{F}\xi\left(\left(\lambda^{kq}\right)_n - \lambda_n\right), u_t - \left(u^{hp}\right)_t\right\rangle_{\Gamma_C}\\
&\leq \left\langle\left(\lambda^{kq}\right)_t - \mathcal{F}\xi\left(\lambda^{kq}\right)_n, u_t - \left(u^{hp}\right)_t\right\rangle_{\Gamma_C} + \mathcal{F}\,\|\xi\|\,\left\|u-u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}\left\|\lambda - \lambda^{kq}\right\|_{\widetilde{H}^{-1/2}(\Gamma_C)}\ .
\end{aligned}$$

For the other term there holds, similarly to [18, Eq. 27],

$$\begin{aligned}
\left\langle\left(\lambda^{kq}\right)_t - \mathcal{F}\xi\left(\lambda^{kq}\right)_n, u_t - \left(u^{hp}\right)_t\right\rangle_{\Gamma_C} &\leq \left\|u-u^{hp}\right\|_{\widetilde{H}^{1/2}(\Gamma_\Sigma)}\left\{\left\|\left(|(\lambda^{kq})_t| - \mathcal{F}(\lambda^{kq})_n^+\right)^+\right\|_{\widetilde{H}^{-1/2}(\Gamma_C)} + \mathcal{F}\,\left\|(\lambda^{kq})_n^-\right\|_{\widetilde{H}^{-1/2}(\Gamma_C)}\right\}\\
&\quad - \left\langle\left(|(\lambda^{kq})_t| - \mathcal{F}(\lambda^{kq})_n^+\right)^-, \left|(u^{hp})_t\right|\right\rangle_{\Gamma_C} + \left\langle\left|(\lambda^{kq})_t\right|, \left|(u^{hp})_t\right|\right\rangle_{\Gamma_C} - \left\langle(\lambda^{hp})_t, (u^{hp})_t\right\rangle_{\Gamma_C}\ .
\end{aligned}$$

This gives the assertion if $\alpha_W - \epsilon_1 - \epsilon_2 - \frac{C}{\beta}\mathcal{F}\,\|\xi\|\,(1+\epsilon_3) > 0$, i.e. $\mathcal{F}\,\|\xi\|$ is sufficiently small.

For a discussion of the assumption $\lambda_t = \mathcal{F}\lambda_n\xi$ where $\xi \in Dir_t(u_t)$, in particular in which cases this assumption cannot be fulfilled, we refer to [19, Remark 2]. That assumption is relaxed in [3] which leads to a very similar a posteriori error estimate.

8. Numerical experiments

For the following numerical experiments we choose $\gamma_0 = 10^{-3}$, $\beta = \eta = 0$ and $\widehat{\mathcal{T}}_k = \mathcal{T}_h$ with $q = p$, i.e. we use the same mesh for λ^{kq} and u^{hp} on Γ_C. Contrary to Remark 17 we do observe measurable algebraic convergence rates, indicating that the a priori error estimate in Theorem 16 may not be sharp. For adaptivity we use the following algorithm with Dörfler marking parameter $\theta = 0.3$ and analyticity parameter $\delta = 0.5$. For δ close to one p-refinements are favored.

We define the local error indicators $\Xi(E)$ of an edge $E \in \mathcal{T}_{h,\Gamma}$ using the right hand side of the a posteriori error estimate in Theorem 22, approximating the dual norm $\|\mu^{kq}\|^2_{\widetilde{H}^{-1/2}}$ by a scaled L^2-norm $kq^{-1}\|\mu^{kq}\|^2_{L^2}$, and $\|v\|^2_{H^{1/2}}$ by $h^{-1}p\|v\|^2_{L^2}$:

$$\begin{aligned}\sum_{E\in\mathcal{T}_{h,\Gamma}} \Xi^2(E) := &\sum_{E\in\mathcal{T}_h|_{\Gamma_N}} \frac{h_E}{p_E}\left\|f - S_{hp}u^{hp}\right\|^2_{L^2(E)} + \sum_{E\in\mathcal{T}_h|_{\Gamma_C}} \frac{h_E}{p_E}\left\|\lambda^{kq} + S_{hp}u^{hp}\right\|^2_{L^2(E)} \\ &+ \sum_{E\in\mathcal{T}_{h,\Gamma}} h_E\left\|\frac{\partial}{\partial s}\left(V\psi^{hp} - (K+\frac{1}{2})u^{hp}\right)\right\|^2_{L^2(E)} + \left\langle\left(\lambda_n^{kq}\right)^+, \left(g - u_n^{hp}\right)^+\right\rangle_{\Gamma_C} + \sum_{E\in\mathcal{T}_h|_{\Gamma_C}} \frac{p_E}{h_E}\left\|\left(g - u_n^{hp}\right)^-\right\|^2_{L^2(E)} \\ &+ \sum_{E\in\mathcal{T}_h|_{\Gamma_C}} \frac{k_E}{q_E}\left\|\left(\lambda_n^{kq}\right)^-\right\|^2_{L^2(E)} + \sum_{E\in\mathcal{T}_h|_{\Gamma_C}} \frac{k_E}{q_E}\left\|\left(\left|\lambda_t^{kq}\right| - \mathcal{F}\right)^+\right\|^2_{L^2(E)} - \left\langle\left(\left|\lambda_t^{kq}\right| - \mathcal{F}\right)^-, \left|u_t^{hp}\right|\right\rangle_{\Gamma_C} - \left\langle\lambda_t^{kq}, u_t^{hp}\right\rangle_{\Gamma_C} + \left\langle\left|\lambda_t^{kq}\right|, \left|u_t^{hp}\right|\right\rangle_{\Gamma_C}.\end{aligned}$$

Algorithm 25. *(Solve-mark-refine algorithm for hp-adaptivity)*

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1. *Choose initial discretization $\mathcal{T}_{h,\Gamma}$ and p, steering parameters $\theta \in (0,1)$ and $\delta \in (0,1)$.*
2. *For $k = 0, 1, 2, \ldots$ do*
   (a) *solve discrete mixed problem* (13).
   (b) *compute local indicators Ξ^2 to current solution.*
   (c) *mark all elements $E \in \mathcal{N} := \operatorname{argmin}\left\{\left|\left\{\hat{\mathcal{N}} \subset \mathcal{T}_{h,\Gamma} : \sum_{E\in\hat{\mathcal{N}}}\Xi^2(E) \geq \theta\sum_{E\in\mathcal{T}_{h,\Gamma}}\Xi^2(E)\right\}\right|\right\}$ for refinement.*
   (d) *estimate local analyticity [22], i.e. compute Legendre coefficients of*

   $$v^{hp}|_E(\Theta_E(x)) = \sum_{j=0}^{p_E} a_i L_i(x), \qquad a_i = \frac{2i+1}{2}\int_{-1}^{1} v^{hp}|_E(\Theta_E(x))L_i(x)\,dx$$

   and use a least-squares approach to compute the slope m of $|\log|a_i|| = mi + b$, for each direction of u^{hp} on Γ_Σ, of ψ^{hp} on Γ_D, respectively. If $e^{-m} \leq \delta$ for all directions, then p-refine, else h-refine marked element E. If $p_E = 0$ always p-refine to have a decision basis next time.
   (e) *refine marked elements based on the decision in 2(d).*

The discrete problems are solved with a semi-smooth Newton method, for which the constraint (13b) is written as two projection equations, one in the normal and one in the tangential component. In all figures we use the abbreviations GLL for Gauss-Lobatto-Lagrange polynomials, GLeL for Gauss-Legendre-Lagrange polynomials and Bernstein for Bernstein polynomials. Each of these three abbreviations states which discrete set, (14) or (12), is used for the discrete Lagrange multiplier.

8.1. Mixed boundary value problem with linear Tresca-friction threshold

For the following numerical experiments, the domain is $\Omega = [-\frac{1}{2}, \frac{1}{2}]^2$ with $\Gamma_C = [-\frac{1}{2}, \frac{1}{2}] \times \left\{-\frac{1}{2}\right\}$, $\Gamma_D = [\frac{1}{4}, \frac{1}{2}] \times \left\{\frac{1}{2}\right\}$ and $\Gamma_N = \partial\Omega \setminus (\Gamma_C \cup \Gamma_D)$. The material parameters are $E = 500$ and $\nu = 0.3$, the gap function is $g = 1 - \sqrt{1 - \frac{x_1^2}{100}}$ and the Tresca friction function is $\mathcal{F} = 0.211 + 0.412x_1$. The Neumann force is

$$f_{\text{left}} = \begin{pmatrix} -(\frac{1}{2} - x_2)(-\frac{1}{2} - x_2) \\ 0 \end{pmatrix} \qquad \text{on } \left\{-\frac{1}{2}\right\} \times \left[-\frac{1}{2}, \frac{1}{2}\right],$$

$$f_{\text{top}} = \begin{pmatrix} 0 \\ 20(-\frac{1}{2} - x_1)(-\frac{1}{4} - x_1) \end{pmatrix} \qquad \text{on } \left[-\frac{1}{2}, -\frac{1}{4}\right] \times \left\{\frac{1}{2}\right\},$$

and zero elsewhere. An example with similar obstacle and friction function is considered in [32] for FEM and in [3] for BEM with biorthogonal basis functions. The solution is characterized by two singular points at the interface from Neumann to Dirichlet boundary condition. These singularities are more severe than the loss of regularity from the contact conditions. At the contact boundary the solution has a long interval in which it is sliding, i.e. where $|\sigma_t| = \mathcal{F}$ and $u_t = -\alpha\sigma_t$ for some $\alpha \geq 0$, and in which the absolute value of the tangential Lagrange multiplier increases linearly like $\mathcal{F}$. The actual contact set, i.e. where $u_n = g$, is slightly to the right of the center of Γ_C.

Figure 1 shows the reduction of the error estimate for different families of discrete solutions. The residual based error estimate for the uniform h-version with $p = 1$ has a convergence rate of 0.5 which is the same as in the non-stabilized case with biorthogonal basis function presented in [3]. Here, the residual contribution of the residual error indicator is divided by a factor of ten to compensate for the residual estimator's typical large reliability constant. This factor is purely heuristic and based on a comparison of the residual and bubble error estimator for contact problems with biorthogonal basis functions in [3]. Employing an h-adaptive scheme improves the convergence rate to 1.6. If both, h- and p-refinements are carried out, the convergence rate is further improved to 1.9. This is a very different behavior to the non-stabilized case with biorthogonal basis functions. There h-adaptivity has a convergence rate of 1.3 and hp-adaptivity of 2.8 and a significant fraction of the adaptive refinements is carried out on the contact boundary Γ_C. In fact, the h-adaptive scheme there shows an almost uniform mesh refinement on a large part of Γ_C which is not that severe here, Figure 2 (a). The reason for that might be that the residual of the variational equation is the dominant contribution of the error indicator, Figure 3. On the contact boundary, this is $\lambda^{kq} + S_{hp}u^{hp}$. However, the employed stabilization tries to achieve that $\lambda^{kq} + S_{hp}u^{hp} = 0$ for each discrete solution. Hence, the estimated error on Γ_C is small and fewer local refinements are carried out there.

Noting that the Bernstein based discretization (12) is the same as the Gauss-Lobatto-Lagrange (GLL) based on (14) if $p = 1$, it is clear that the error estimate does not differ between these two approaches for both the uniform and the h-adaptive scheme, Figure 1. Even though, the constraints in the Gauss-Legendre-Lagrange (GLeL) approach are different then in the other two approaches, and in particular the GLeL approach is also non-conforming even for $p = 1$, there is no significant difference in the error estimate, expect in the preasymptotic range. When looking at the hp-refined meshes for these three approaches, Figure 2 (b)-(d), it becomes clear why the difference in the error estimates is that small. Nevertheless, in the GLeL approach the consistency errors in λ_n and λ_t are non-zero, Figure 3(c)-(d), contrary to the conforming Bernstein polynomial case, Figure 3(b). The error contributions for the GLL approach are almost identical to the Bernstein polynomial case and are therefore omitted here.

8.2. Neumann boundary value problem with Coulomb-friction

For the following numerical experiments, the domain is $\Omega = [-\frac{1}{2}, \frac{1}{2}]^2$ with $\Gamma_C = [-\frac{1}{2}, \frac{1}{2}] \times \left\{-\frac{1}{2}\right\}$ and $\Gamma_N = \partial\Omega \setminus \Gamma_C$. Since no Dirichlet boundary is prescribed, the kernel of the Steklov operator consists of the three rigid body motions $\ker(S) = \operatorname{span}\{(x_1, 0)^\top, (0, x_2)^\top, (x_2, -x_1)^\top\}$. Nevertheless, to obtain a unique solution the rigid body motions are forced to zero during the simulation. The material parameters are $E = 5$ and $\nu = 0.45$, and the Coulomb friction coefficient is 0.3. The Neumann force is

$$f_{\text{side}} = \begin{pmatrix} -10\,\operatorname{sign}(x_1)(\frac{1}{2} + x_2)(\frac{1}{2} - x_2)\exp(-10(x_2 + \frac{4}{10})^2) \\ \frac{7}{8}(\frac{1}{2} + x_2)(\frac{1}{2} - x_2) \end{pmatrix},$$

$$f_{\text{top}} = \begin{pmatrix} 0 \\ -\frac{25}{2}(\frac{1}{2} - x_1)^2(\frac{1}{2} + x_1)^2 \end{pmatrix}$$

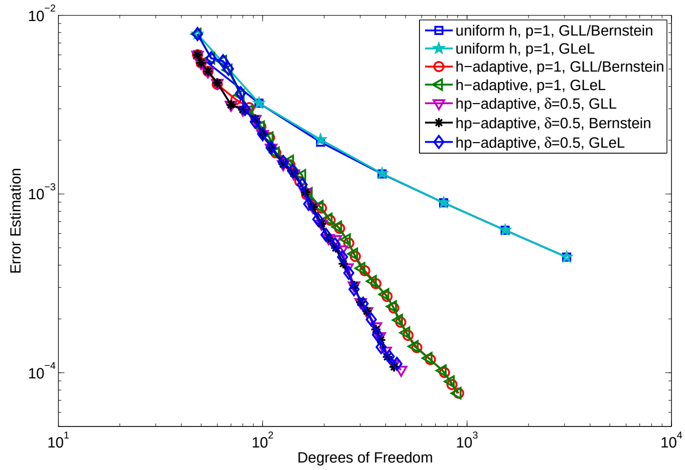


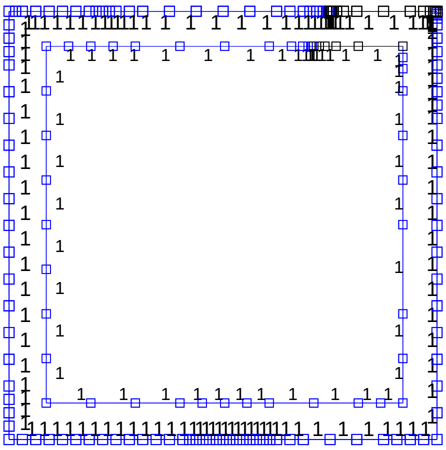
Figure 1: Error estimates for different families of discrete solutions (Tresca-friction)

on the side, respectively on the top, and the gap to the obstacle is zero. A similar example is considered in [23] for FEM and the same in [3] for BEM with biorthogonal basis functions. The solution is characterized by a large contact set and that the Lagrange multiplier has a kink, jump in the normal, tangential component, respectively, at $x_1 = 0$, Figure 4.

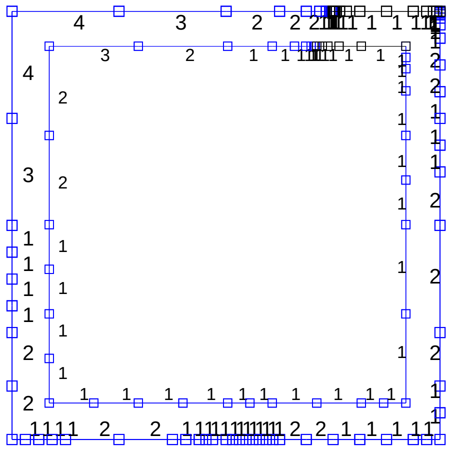
Figure 5 shows the reduction of the error estimate for different families of discrete solutions. The residual based error estimate for the uniform h-version with $p = 1$ has an optimal convergence rate of almost 1.5. Thus, employing an h-adaptive scheme improves the convergence rate in the preasymptotic range but then the estimated error runs parallel to the uniform case as only quasi uniform mesh refinements are carried out, Figure 6 (a). If both, h- and p-refinements are carried out, the convergence rate is improved to over 2.8 after a preasymptotic range in which only h-refinements are been carried out. The estimated error for the GLL- and Bernstein approach is the same even for the hp-adaptive case, since the basis functions for the Lagrange multiplier and the contact conditions only differ where $p \geq 2$. This however, is only the case outside the actual contact area, Figure 6 (b)-(c), but there $\lambda = 0$ due to Coulomb's friction law. The estimated error for the GLeL approach does not differ to the other two approaches in a significant manner, neither in the asymptotic nor in the preasymptotic range. The error reduction and adaptivity behavior is again very different to the non-stabilized case with biorthogonal basis function [3, Sec. 6.2]. There the convergence rate is larger with 1.9 for h-adaptivity and 3.3 for hp-adaptivity and the refinements on Γ_C are more localized. There, the dominant error source is the stick-slip contribution, and thus explaining the local mesh refinements on Γ_C, whereas here the residual of the variational equation and the violation of the complementarity condition in λ_n are dominant. Interestingly, here, the stick-slip contribution is the smallest non-zero error contribution and is several orders of magnitudes smaller then the other remaining ones, Figure 7.

8.3. *Influence of the stabilization for the Neumann boundary value problem with Coulomb-friction*

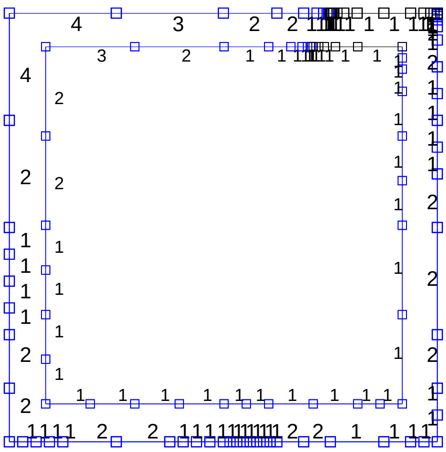
From Lemma 6 it is clear that if γ_0 is chosen to be too large the system matrix has at least one negative eigenvalue and the entire theory may no longer hold. For a uniform mesh with 256 elements and $p = 1$ the dependency of the error estimate on the parameter γ_0 is displayed in Figure 8. In all cases the iterative solver converges to a solution of the discrete problem. But for $\gamma_0 \geq 0.152$ the system matrix has a negative eigenvalue and the discrete solution looks unphysical or even simply useless. Interestingly, the error estimate captures this partly, the red curve in Figure 8, even though the error estimate may not be an upper bound of the discretization error. Once γ_0 is sufficiently small, here $1.9 \cdot 10^{-12} \leq \gamma_0 \leq 6.6 \cdot 10^{-2}$, there is (almost) no dependency on the absolute value of γ_0 itself, neither in the error estimate nor in the discrete solution itself. Only if γ_0 is further decreased, i.e. the stabilization is effectively switched off, the Lagrange multiplier in the GLL/Bernstein approach starts to oscillate as it is typical for the non-stabilized case, when using the same mesh and polynomial degree for $u^{hp}|_{\Gamma_C}$ and λ^{kq} and no special basis functions. This is captured by the increase in the error estimate. Interestingly, in the GLeL approach, the Lagrange multiplier almost



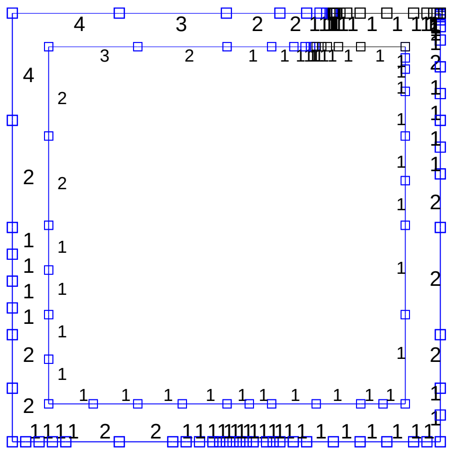
(a) h-adaptive (GLL/Bernstein), mesh nr. 10 (inner), nr. 20 (outer)



(b) hp-adap. (Bernstein), mesh nr. 10 (in), nr. 20 (out)

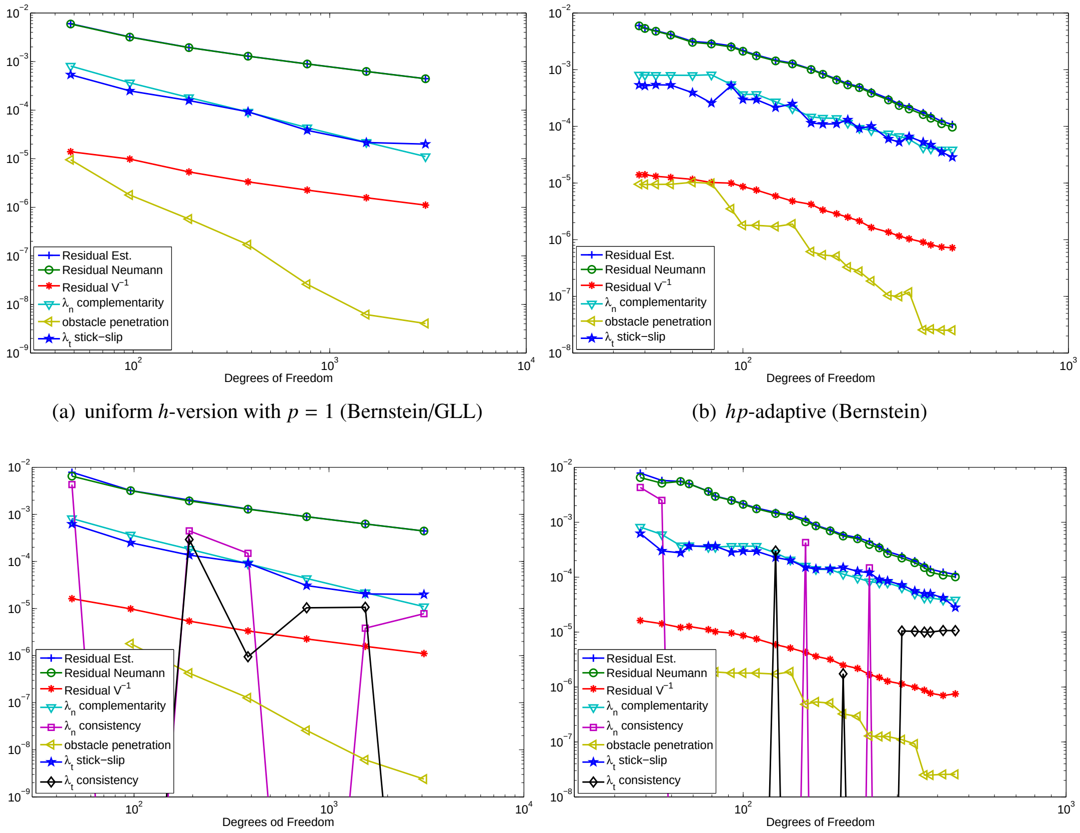


(c) hp-adap. (GLL), mesh nr. 10 (inner), nr. 20 (outer)



(d) hp-adap. (GLeL), mesh nr. 10 (inner), nr. 20 (outer)

Figure 2: Adaptively generated meshes (Tresca-friction)



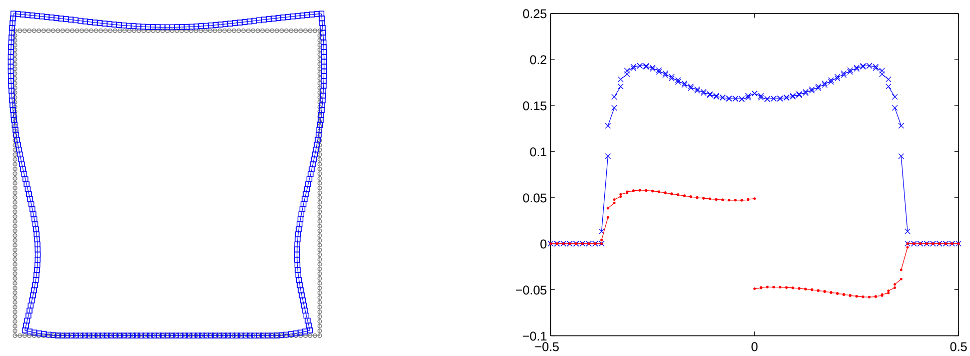
(a) uniform h-version with $p = 1$ (Bernstein/GLL)

(b) hp-adaptive (Bernstein)

(c) uniform h-version with $p = 1$ (GLeL)

(d) hp-adaptive (GLeL)

Figure 3: Error contributions of the residual based error estimate (Tresca-friction)



(a) Reference (circle), deformed (square)

(b) λ_n (cross), λ_t (dot)

Figure 4: Solution of the Coulomb-frictional problem, uniform mesh 256 elements, $p = 1$ (GLL/Bernstein)

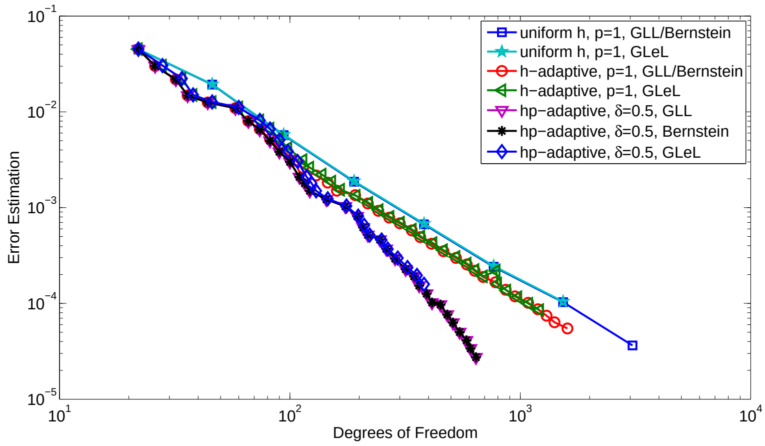
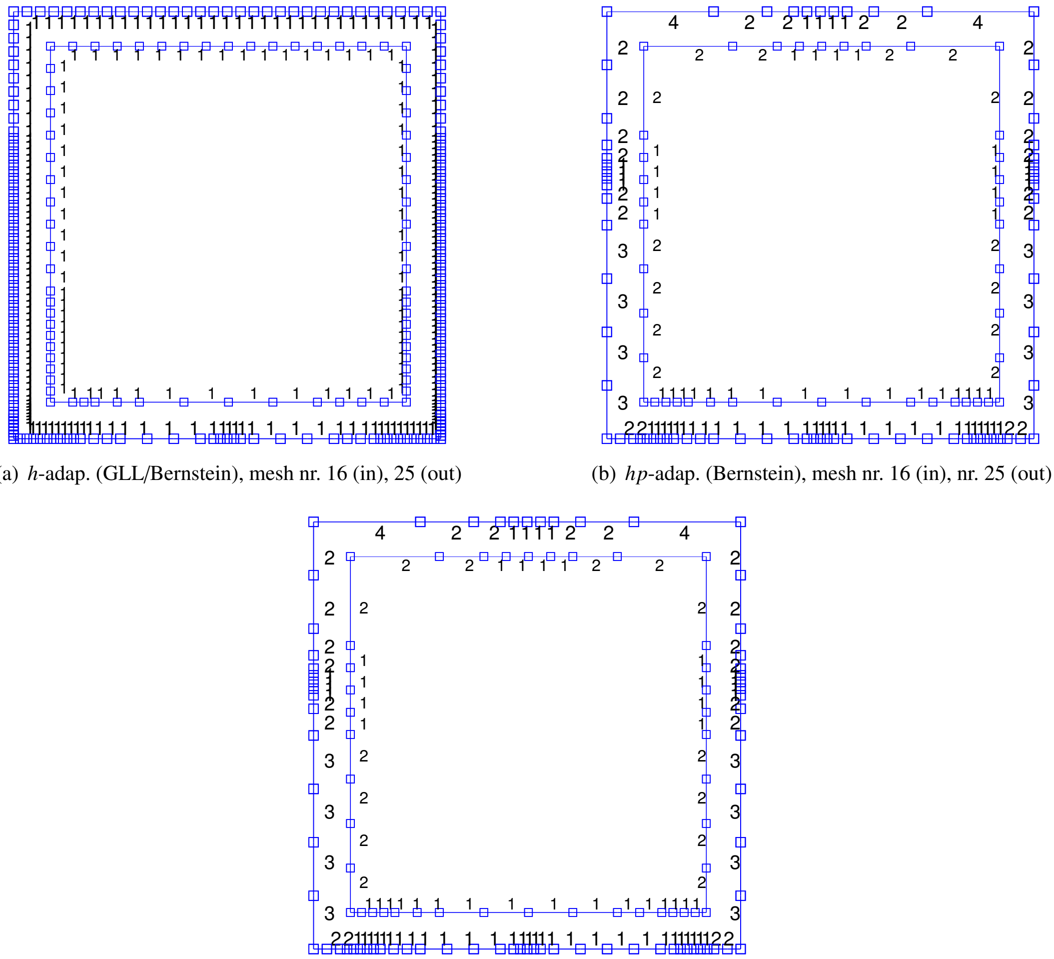


Figure 5: Error estimates for different families of discrete solutions (Coulomb-friction)

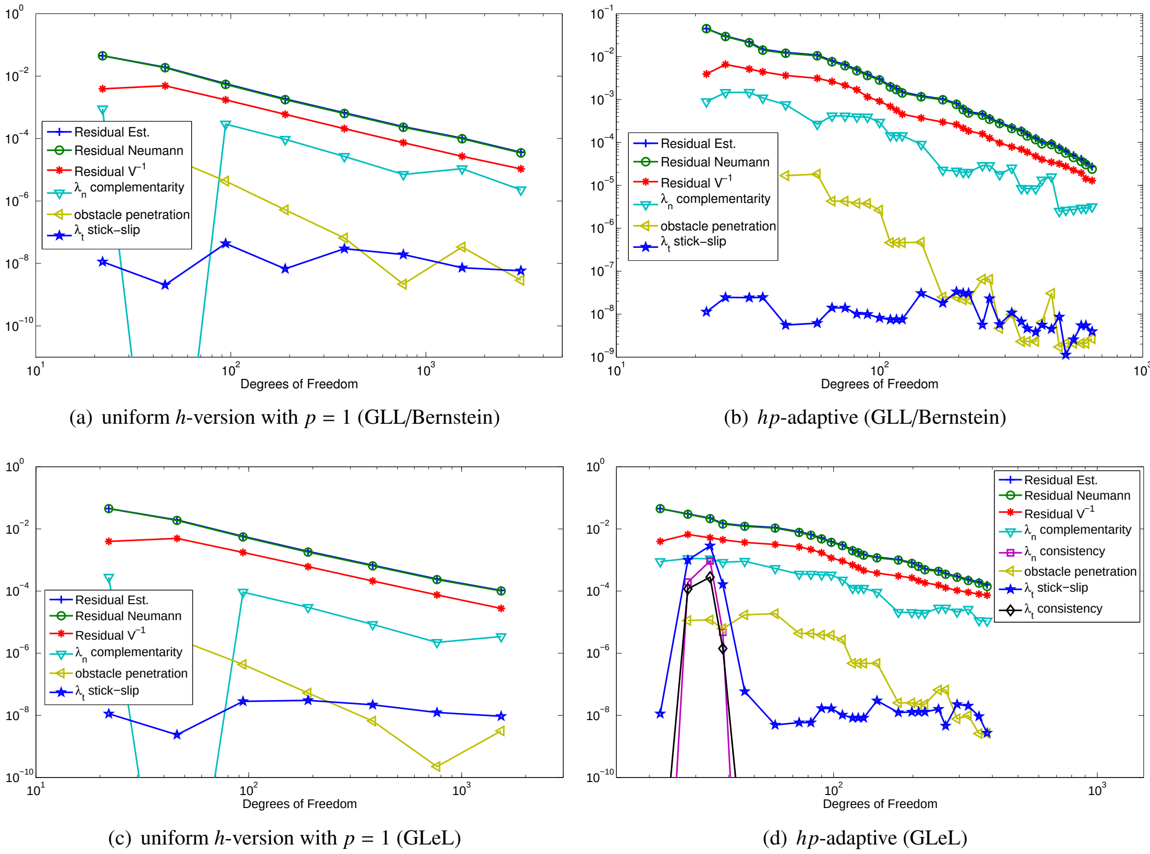


(a) h-adap. (GLL/Bernstein), mesh nr. 16 (in), 25 (out)

(b) hp-adap. (Bernstein), mesh nr. 16 (in), nr. 25 (out)

(c) hp-adap. (GLL), mesh nr. 16 (inner), nr. 25 (outer)

Figure 6: Adaptively generated meshes (Coulomb-friction)



(a) uniform h-version with $p = 1$ (GLL/Bernstein)

(b) hp-adaptive (GLL/Bernstein)

(c) uniform h-version with $p = 1$ (GLeL)

(d) hp-adaptive (GLeL)

Figure 7: Error contributions of the residual based error estimate (Coulomb-friction)

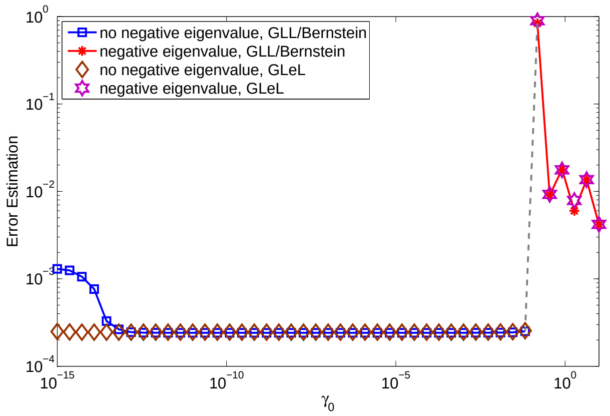


Figure 8: Dependency of error estimate on γ_0 for uniform mesh with 256 elements and $p = 1$ (Coulomb-friction)

does not oscillate for extremely small γ_0, which is reflected by the error estimate.

Within the simulation, the most time consuming contribution is the computation of the matrices $\widehat{\mathbf{WW}}$, $\widehat{\mathbf{K}^\top\mathbf{K}^\top}$ and $\widehat{\mathbf{WK}^\top}$ for the stabilization matrix $\widehat{\mathbf{S}}$. Since γ_0 is allowed to be very small it may be favorable to compute these matrices only approximately. In the following we replace $\widehat{\mathbf{WW}}$ by $\overline{\mathbf{W}}^\top\mathbf{M}_D^{-\top}\mathbf{M}_\gamma\mathbf{M}_D^{-1}\overline{\mathbf{W}}$, where

$$\left(\mathbf{M}_\gamma\right)_{i,j} := \left\langle \gamma\phi_j, \phi_i\right\rangle_{\Gamma_C}, \quad (\mathbf{M}_D)_{i,j} := \left\langle \phi_j, \phi_i\right\rangle_{\Gamma_\Sigma}, \quad \left(\overline{\mathbf{W}}\right)_{i,j} := \left\langle W\varphi_j, \phi_i\right\rangle_{\Gamma_\Sigma}, \quad \left(\overline{\mathbf{K}}^\top\right)_{i,j} := \left\langle K^\top\phi_j, \phi_i\right\rangle_{\Gamma_\Sigma}, \tag{42}$$

with $\operatorname{span}\{\phi_i\}_i = \mathcal{V}^D_{hp+1}$ and $\operatorname{span}\{\varphi_i\}_i = \mathcal{V}_{hp}$. In particular $\mathbf{M}_D$ is only a block-diagonal matrix and thus its inverse is cheap. The difference to the original formulation in Section 6 is in an intermediate projection of Wu^{hp}, Wv^{hp} onto the discontinuous finite element space $\mathcal{V}^D_{hp+1}$. Analogously, the matrices $\widehat{\mathbf{K}^\top\mathbf{K}^\top}$, $\widehat{\mathbf{WK}^\top}$ are replaced by $(\overline{\mathbf{K}}^\top)^\top\mathbf{M}_D^{-\top}\mathbf{M}_\gamma\mathbf{M}_D^{-1}\overline{\mathbf{K}}^\top$, $(\overline{\mathbf{K}}^\top)^\top\mathbf{M}_D^{-\top}\mathbf{M}_\gamma\mathbf{M}_D^{-1}\overline{\mathbf{W}}$, respectively. Even though four instead of three matrices must now be computed, only two potentials (due to element-wise integration by parts for W) must be evaluated and thus this is significantly faster.

Figure 9 shows the decay of the error estimate for the uniform h version with $p = 1$ and for the hp-adaptive scheme with Gauss-Lobatto-Lagrange basis functions when using the above approximation of the stabilization matrix. For comparison the corresponding curves from Figure 5 are also depicted. The difference in the error estimate for the original stabilization approach and its approximation is $\pm 0.014\%$ for the uniform h-version with $p = 1$ and $\pm 0.02\%$ for the hp-adaptive scheme.

The analysis of this approximate stabilization, as well as simpler stabilizations which are not based on S^2_{hp}, are left for future work.

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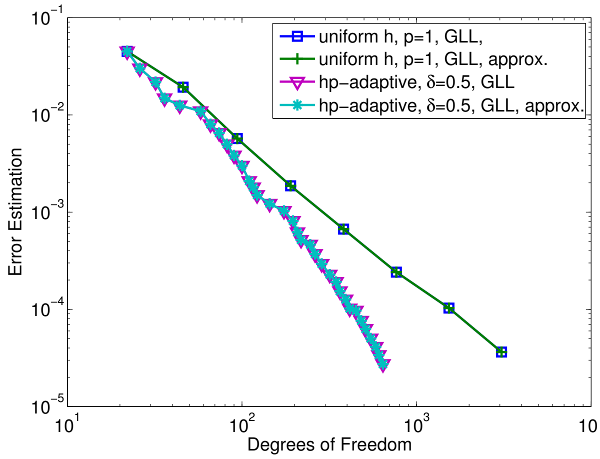

Figure 9: Error estimate for families of discrete solutions with and without approximation of the stabilization matrix $\widehat{\mathbf{S}}$ (Coulomb-friction)

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